CHAPTER 10

DIFFERENTIAL EQUATIONS

10.1 Introduction

We began the study of dynamic analysis with an introduction to integral calculus in Chapter 8. We discussed there the topics such as indefinite integrals, definite integrals improper integrals, and some of their applications. Our most important aim in integral calculus was to derive the primitive function from the derived function. As mentioned at the beginning of Chapter 9, the reader must have noticed an important point there: the independent variable of the derived function (and, hence, the primitive function) may or may not be a function of time. In short, in integral calculus we were mainly interested in determining the primitive function using the derived function, which may or may not be a function of time.

However, we shifted our attention in Chapter 9 to functions that were defined by time. We understood from that chapter that many of the relationships in the subjects of our interest were depended on time such that the value of one variable in a particular time period was a function of the value(s) of the same or other variables in the past time period(s). In these relationships we assumed that time was changing discretely. Then one important question we wanted to answer in that chapter was how one could determine the path of the variable defined discretely by time. Another important question was to determine whether this path tended to a steady-state value when time assumed a sufficiently large value or in the long-run. To answer these and related questions, we introduced and applied an important branch of mathematics called difference equation.

We mentioned at the end of Section 9.1 that one could find in the subjects of our interest that relationships depended upon time not just discretely, but continuously too. We referred to an analysis of variables that depended upon time when time is considered to be continuous as continuous time analysis. We also stated there that an important mathematical technique of continuous time analysis was differential equation. Therefore, in the present chapter we turn our attention to the basics of a branch of mathematics called differential equations.

As mentioned above, our main aim in the last chapter was to determine the time paths of discrete variables or solve the difference equations and determine whether their solutions would converge to or diverge from steady-state or long-run equilibrium, either identical or distinct. The reader must have noticed that the said solutions were not particular values, but were discrete functions of time. Our aim in the present chapter is similar: determine the time paths of continuous variables or solve differential equations and determine whether these time paths or solutions converge to or diverge from steady-state or long-run equilibrium, again either identical or distinct. Again, these solutions will not be particular values, but will be continuous functions of time. Moreover, as we will see through the following sections, there exists a close resemblance between difference and differential equations with regard to the methods of solutions and convergence analyses. Therefore, our study of difference equations can lead us, up to a great extent, in our attempt to study differential equations.

10.2 First-Order Differential Equations

10.2.1 Definitions, notations and classifications

Let us first define a differential equation. A differential equation is defined as an equation that states a functional relationship between a dependent variable and time (as the independent variable) and one or more derivatives or differentials of this function. Notice that this definition contains two important components. Firstly, a differential equation is an equation that involves an unspecified function such as $y = f(t)$. Secondly, a differential equation involves one or more of the derivatives or differentials of this function.

We shall now present few examples of differential equations. For simplicity, we shall follow our exposition in the last chapter. Suppose that the rate of change of price (y) with respect to time (t) is a function of the price itself. We may write this function as

$$dy(t)/dt = \dot{y}(t) = ay(t) + b, \tag{10.2.1}$$

where a and b are constants, a denotes the coefficient and b denotes the term, and the dot over $y(t)$ shows that it is the derivative of the function $y(t)$ with respect to time: $dy(t)/dt = \dot{y}(t)$. Notice that equation (10.2.1) contains both the term $dy(t)/dt$ and the term $y(t)$. Equations such as (10.2.1) are called differential equations. If we assume that $dy(t)/dt$ is also a direct function of time, then equation (10.2.1) may be written as

$$dy(t)/dt = \dot{y}(t) = ay(t) + bt. \tag{10.2.2}$$

As another example, suppose that the rate of change of the rate of change of price with respect to time is a function of both the rate of change of price and price as such, then we can write the function as a differential equation given by

$$d^2y(t)/dt^2 = \ddot{y}(t) = a_1[dy(t)/dt] + a_2y(t) + b = a_1\dot{y}(t) + a_2y(t) + b. \quad (10.2.3)$$

If we assume that the rate of change of the rate of change of price is directly influenced by time, then the last differential equation may be written as

$$d^2y(t)/dt^2 = \ddot{y}(t) = a_1[dy(t)/dt] + a_2y(t) + bt = a_1\dot{y}(t) + a_2y(t) + bt. \quad (10.2.4)$$

Suppose, instead of the forms in equations (10.2.3) and (10.2.4), that the rate of change of price is given by

$$dy(t)/dt = \dot{y}(t) = ay(t)^{\alpha} + b, \quad (10.2.5)$$

which also is a differential equation. If we assume that the rate of change of price is also affected by time directly, then differential equation (10.2.5) may be written as

$$dy(t)/dt = \dot{y}(t) = ay(t)^{\alpha} + bt. \quad (10.2.6)$$

Lastly, suppose that the rate of change of the rate of change of price with respect to time is a function of both the rate of change of price and price as such, as a function of time, then we may write [instead of the form in equation (10.2.5)] the function as a differential equation as

$$d^2y(t)/dt^2 = \ddot{y}(t) = a_1[dy(t)/dt] + ay(t)^{\alpha} + b = a_1\dot{y}(t) + ay(t)^{\alpha} + b; \quad (10.2.7)$$

or, if $d^2y(t)/dt^2$ is directly affected by time, as

$$d^2y(t)/dt^2 = \ddot{y}(t) = a_1[dy(t)/dt] + ay(t)^{\alpha} + b = a_1\dot{y}(t) + ay(t)^{\alpha} + bt. \quad (10.2.8)$$

As in the case of difference equations, let us now present the **classification of differential equations** found in the literature. One important classification is based on the order with which they appear. The order of a differential equation is determined by the highest order of the derivative(s) in that equation. Consider first equation (10.2.1): $dy(t)/dt = ay(t) + b$. Notice that the highest derivative in this equation is the first-order derivative $dy(t)/dt$ and, therefore, it is called a **first-order differential equation**. Similarly, equations (10.2.2), (10.2.5) and (10.2.6) are first-order differential equations. But the highest order of the derivatives in equations (10.2.3), (10.2.4), (10.2.7) and (12.0.8) is 2 and, therefore, they are **second-order differential equations**. In general, if the highest order of the derivative(s) in a differential equation is n, then that differential equation is called a **n^{th}-order differential equation**.

The next important classification of differential equations is based on whether they are linear or nonlinear. Consider equation (10.2.1). In this equation the term $dy(t)/dt$ or the term $y(t)$ appears with power one or degree one. Moreover, these terms are not multiplied or divided by any other derivative of y with respect to t or by the term $y(t)$. Therefore, this equation is called a **linear differential equation**. In general, if a differential equation contains derivatives and dependent variable that are raised to power 1 and the derivatives or the dependent variable are not multiplied or divided by any other derivatives or variable, then that equation is called a linear differential equation; otherwise, it is called a **nonlinear differential equation**. Following this definition, we see that equations (10.2.1) through (10.2.4) are linear differential equations and equations (10.2.5) through (10.2.8) are nonlinear difference equations if $\alpha \neq 1$.

The last classification of differential equations we make here is based on the nature of the coefficients ($a's$) and terms ($b's$) in the equations. If the coefficient a and the term b in a differential equation are constants, then it is called **autonomous differential equations**. Applying this definition, we see that equations (10.2.1), (10.2.3), (12.0.5), and (10.2.7) are autonomous differential equations. If in a differential equation either the coefficient a or the term b (or both) is (are) some functions of time, then the equation is called a non-autonomous differential equation. Therefore, equations (10.2.2), (10.2.4), (10.2.6), and (10.2.8) are non-autonomous differential equations. If in a differential equation the term b is zero ($b = 0$), then that equation is called a **homogenous differential equation**; and if the term b is not equal to zero ($b \neq 0$), then that equation is called a **non-homogenous differential equation**.

10.2.2 Solution of first-order, linear, autonomous differential equations

Suppose that we have a differential equation similar to the one in equation (10.2.1): $dy(t)/dt = ay(t) + b$. The reader must have noticed that this is a first-order, linear, autonomous differential equation. As stated earlier, our aim is to determine the time path of the variable $y(t)$; that is, to solve the differential equation. Solving a differential equation means determining the underlying function of time that generated the equation or that makes the equation true.

We know from Chapter 9 that the solution to a difference equation comprises of two components: the complementary function and the particular solution. The complementary function represents the solution to the homogenous form of the equation and the particular solution represents any solution that satisfies the equation. Recall that the particular solution gives us the steady-state value (or the intertemporal equilibrium value) to which or from which the complementary function may tend as time increases. The solution to a differential equation also comprises of the same components and their roles are also the same as those in the solution to a difference equation.

Consider the equation $dy(t)/dt = ay(t) + b$. First of all, let us attempt to find the complementary function of the solution to this equation. For this we shall work with the homogenous form of the equation by setting its term to zero: $dy(t)/dt = ay(t)$. Notice that the last equation may be written in an alternative form as $[1/y(t)]dy(t) = adt$. Let us now apply the integral sign on both sides of the equation to obtain $\int [1/y(t)]dy(t) = \int adt$. Applying the rules of integration, the last equation can be written as $\ln y(t) + C_1 = at + C_2$ or $\ln y(t) = at + C_2 - C_1$. This result can be simplified, using the properties of logarithm, to obtain $e^{\ln y(t)} = e^{(c_2 - c_1)} e^{at}$, or $y(t) = e^{(c_2 - c_1)} e^{at}$. Assuming that $C = e^{(c_2 - c_1)}$, the last equation can be written as

$$y(c) = Ce^{a\,t}, \tag{10.2.9}$$

where $y(c)$ shows that the solution represents the complementary function. Notice the similarities between this solution and the solution to the difference equation presented in Section 9.2.3 [the part $y_c = Ca^t$ in equation (9.2.18)]. Firstly, in both solutions we have a constant C, which can be definitized using initial conditions. Secondly, in both solutions we have time as an exponent. Thirdly, the role of the coefficient has been reversed: in the former it has become as an exponent while in the latter it was the base of the exponential term. Lastly, the former solution differs from the latter due to the presence of e, the base of natural logarithm, as the base of the exponential term. Notice also that, as in the case of solutions to difference equations, the complementary function is a function of time.

Let us now attempt to determine the particular solution, which is to do with the complete, non-homogenous differential equation (10.2.1) and not its homogenous form. We know from Chapter 9 that the particular solution is any solution that satisfies the equation. We used there the simplest possible solution, a constant, as the particular solution. We can follow the same procedure to find the particular solution to equation (10.2.1). Suppose that our constant solution is $y(t) = p$. Since p is a constant, its time derivative must be zero; that is, $dy(t)/dt = d(p)/dt = 0$. Substitution of these values into the differential equation $dy(t)/dt = ay(t) + b$ yields $0 = ap + b$, or $p = -b/a$. Therefore, the particular solution in our present case is

$$y(s) = p = -(b/a), \tag{10.2.10}$$

where $a \neq 0$. If $a = 0$ our trial solution will not work and, therefore, we will have to use trial solutions (as in the case of difference equations) such as $y(t) = pt$, $y(t) = pt^2$, etc.

Having obtained both the complementary function [(equation (10.2.9)] and the particular solution [equation (10.2.10)], we can now combine them to obtain the complete solution to the differential equation $dy(t)/dt = ay(t) + b$ as

$$y(t) = y(c) + y(s) = Ce^{at} - (b/a). \tag{10.2.11}$$

We know that the complete solution to the equation $dy(t)/dt = ay(t) + b$ in (10.2.11) is not definite due to the presence of the constant C. We can now definitize the solution using initial condition. Suppose that the value of y when $t = 0$ is $y(0)$. Therefore, substituting $y(0)$ for y when $t = 0$ into equation (10.2.11) yields $y(0) = Ce^{a\times 0} - (b/a) = C - (b/a)$, or $C = y(0) + (b/a)$. Substitution of this result for the constant C into equation (10.2.11) gives

$$y(t) = y(c) + y(s) = \left[y(0) + (b/a)\right]e^{at} - (b/a), \tag{10.2.12}$$

which is the complete and definite solution to the differential equation $dy(t)/dt = ay(t) + b$.

We shall now check whether the solution in equation (10.2.12) is correct or not. For this let us first differentiate the solution with respect to time and obtain $dy(t)/dt = [y(0) + (b/a)].a.e^{at}$. The next step is to substitute the last equation for $dy(t)/dt$ and the solution in equation (10.2.12) for $y(t)$ into the differential equation $dy(t)/dt = ay(t) + b$. The result will be $[y(0) + (b/a)].a.e^{at} = [y(0) + (b/a)].a.e^{at} - (b/a).a + b$, or $[y(0) + (b/a)].a.e^{at} = [y(0) + (b/a)].a.e^{at}$. This implies that, if the initial values are equal, the solution is correct.

As an example, consider the equation $dy(t)/dt = 0.5y(t) + 1$. Notice that this is a linear, first-order, autonomous, non-homogenous differential equation and is identical with equation (10.2.1) for $a = 0.5$ *and* $b = 1$. This implies that we can apply equation (10.2.11) to find the complete solution. The complementary function can be found by applying equation (10.2.9): $y(c) = Ce^{at} = Ce^{0.5t}$. The particular solution can be obtained by applying equation (10.2.10): $y(s) = -b/a = -1/0.5 = -2$. We can now apply equation (10.2.11) to obtain the complete solution: $y(t) = y(c) + y(s) = Ce^{at} - b/a = Ce^{0.5t} - 1/0.5 = Ce^{0.5t} - 2$. Let us now use the initial condition that $y(0) = 1$. Then, from the last equation, we obtain that $y(0) = C - 2 = 1$ or $C = 3$. Thus, the complete and definite solution can be written as $y(t) = 3e^{0.5t} - 2$. The graph of this solution is illustrated in Panel (A) of Figure 10.2.1.

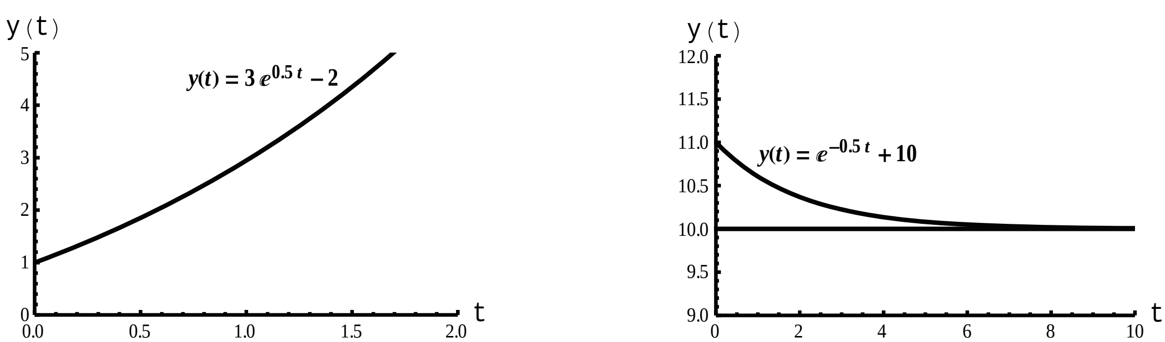


Figure 10.2.1

We shall now check to see whether the solution, $y(t) = 3e^{0.5t} - 2$, we obtained above is correct or not. The derivative of the solution with respect to time yields $dy(t)/dt = 1.5e^{0.5t}$. We can now substitute the derivative and the solution into the differential equation to obtain $dy(t)/dt = 0.5(3e^{0.5t} - 2) + 1 = 1.5e^{0.5t} - 1 + 1 = 1.5e^{0.5t}$. Since the differential equation is identical with the derivative of the solution, the solution is correct.

As another example, consider the equation $dy(t)/dt = -0.5y(t)+5$. Notice that this is a linear, first-order, autonomous, non-homogenous differential equation and is identical with equation (10.2.1) for $a = -0.5$ *and* $b = 5$. The complementary function can be found by applying equation (10.2.9): $y(c) = Ce^{at} = Ce^{-0.5t}$. The particular solution can be obtained by applying equation (10.2.10): $y(s) = -b/a = -5/-0.5 = 10$. We can now apply equation (10.2.11) to obtain the complete solution: $y(t) = y(c)+y(s) = Ce^{at} - b/a = Ce^{-0.5t} - 5/-0.5 = Ce^{-0.5t} + 10$. Let us now definitize the solution with the initial condition $y(0) = 11$. With this condition, we obtain $C = 1$. Therefore, the definite and complete solution can be written as $y(t) = e^{-0.5t} + 10$, the graph of which is illustrated in Panel (B) of Figure 10.2.1.

As in the last example, let us check whether the solution we obtained is correct or not. The derivative of the solution with respect to time is $dy(t)/dt = -0.5e^{-0.5t}$. We can now substitute the solution into the differential equation to obtain $dy(t)/dt = -0.5(e^{-0.5t}+10)+5 \quad = -0.5e^{-0.5t} - 5 + 5 = -0.5e^{-0.5t}$. Since the differential equation is identical with the derivative of the solution, the solution is correct.

10.2.3 Stability, steady-state and convergence

In Section 9.2.4 we presented the topics of stability, steady-state and convergence with regard to first-order, linear, autonomous difference equations. We inferred there that the solution to such a difference equation would converge to (diverge from) the steady-state value y_s, which was nonzero when $b \neq 0$ and was zero when $b = 0$, if the absolute value of the coefficient of the equation, a, was less (greater) than one. One important question now is whether one can use such a condition to determine convergence and divergence in the case of first-order, linear, autonomous differential equations. The answer is yes; although the condition in the case of differential equations is the sign of the coefficient a, and not its absolute value, as we see below.

Consider, for example, the definite and complete solution in equation (10.2.12) to the equation (10.2.1): $dy(t)/dt = ay(t)+b$. This solution is $y(t) = y(c)+y(s) = [y(0)+(b/a)]e^{at} - b/a$. We are interested in finding the value of this solution as time increases; that is, as $t \to \infty$. To do this we take the limit of the last equation as $t \to \infty$: $\lim_{t\to\infty} y(t) = \lim_{t\to\infty}\{[y(0)+(b/a)]e^{at} - b/a\}$. Since both a *and* b are constants, the limit on the RHS of the last equation is determined solely by the sign of the constant coefficient a. If $a > 0$, the RHS of the last equation will tend to infinity (positive infinity if $[y(0)+b/a] > 0$ and negative infinity if $[y(0)+b/a] < 0$). On the other hand, if $a < 0$, the RHS of the last equation will tend to $(-b/a)$.

Therefore, whether the solution to an equation such as (10.2.1) converges to, or diverges from, the particular solution or the steady-state value of $(-b/a)$ depends upon the sign of the coefficient a of the equation. If $a > 0$, the solution or the path of $y(t)$ will diverge from the steady-state value $(-b/a)$; and if $a < 0$, the solution or the path of $y(t)$ will converge to the steady-state value $(-b/a)$. These two cases are illustrated through the graphs in Panels (A) and (B) respectively of Figure 10.2.1.

Notice that the steady-state value will coincide with a point on the horizontal axis if $b = 0$ and it will be equal to $(-b/a)$ if $b \neq 0$. Therefore, our reference point of convergence or divergence in the case of differential equations is the particular solution or the steady-state value $(-b/a)$. Notice also that the reference point of convergence and divergence in the case of first-order, linear, autonomous difference equation, as given in equation (10.2.18), was $b/(1-a)$.

As an example, consider the equation $dy(t)/dt = 0.75y(t)+7.5$, which is a first-order, linear, autonomous, non-homogenous differential equation and is identical with equation (10.2.1) for $a = 0.75$ *and* $b = 7.5$. The complementary function and the particular solution can be found by applying equations (10.2.9) and (10.2.10), respectively. Thus, the complementary function can be written as $y(c) = Ce^{at} = Ce^{0.75t}$ and the particular solution can be written as $y(s) = -b/a = -7.5/0.75 = -10$. Now applying equation (10.2.11) we can obtain the complete solution $y(t) = y(c) + y(s) = Ce^{at} - b/a = Ce^{0.75t} - 10$. With the initial condition $y(0) = -5$, we obtain $C = 5$; and, therefore, the complete and definite solution is $y(t) = 5e^{0.75t} - 10$. Since $a = 0.75 > 0$ in the solution, the results above suggests that the solution $y(t)$ will diverge from the steady-state value -10. This is illustrated in Panel (A) of Figure 10.2.2. Notice that the graph in this figure is similar to the graph in Panel (A) of Figure 10.2.1.

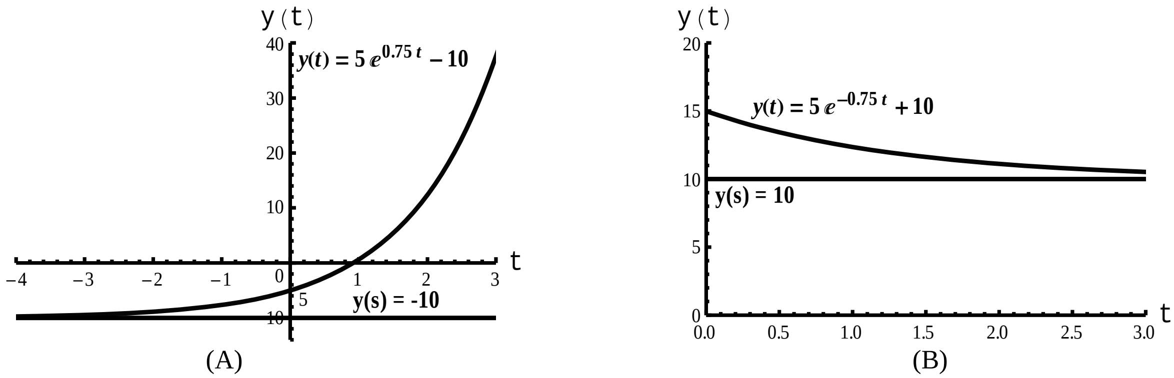


Figure 10.2.2

As another example, consider the equation $dy(t)/dt = -0.75y(t) + 7.5$, which is a first-order, linear, autonomous, non-homogenous differential equation and is identical with equation (10.2.1) for $a = -0.75$ *and* $b = 7.5$. The complementary function and the particular solution can be found by applying equations (10.2.9) and (10.2.10) respectively. Thus, the complementary function can be written as $y(c) = Ce^{at} = Ce^{-0.75t}$ and the particular solution can be written as $y(s) = -b/a = -7.5/-0.75 = 10$. Now applying equation (10.2.11) we obtain the complete solution $y(t) = y(c) + y(s) = Ce^{at} - b/a = Ce^{-0.75t} + 10$. Definitizing the solution with the initial condition $y(0) = 15$, we obtain $C = 5$. Therefore, the complete and definite solution is $y(t) = 5e^{-0.75t} + 10$. Since $a = -0.75 < 0$ in the last equation, our results above suggests that the solution $y(t)$ will converge to the steady-state value 5. This is illustrated in Panel (B) of Figure 10.2.2. Notice that the graph in this figure is similar to the graph in Panel (B) of Figure 10.2.1.

So far we were attempting to determine the path of the solution $y(t) = Ce^{at} - b/a$ or $y(t) = [y(0) + b/a]e^{at} - b/a$, where $C = y(0) + b/a$, in equations (10.2.11) or (10.2.12), respectively. We presented the graphs of the solutions to two differential equations in Figure 10.2.2 with $C = 5$. One important question is: what happens to the solution paths when $C \neq 5$. As in the case of difference equations, we will obtain both scale effect and mirror effect due to the size and sign, respectively, of C. The scale effect will push the path up or down the paths depending upon the size of C and the mirror effect will provide us with opposite paths.

As an example, consider the solution $y(t) = 5e^{0.75t} - 10$ to the equation $dy(t)/dt = 0.75y(t) + 7.5$ of the first example in the present section. We illustrated the graph of this solution in Panel (A) of Figure 10.2.2 for $C = 5$. Suppose now that $C = 10$ and $C = -10$. With these values, we obtain the solutions $y(t) = 10e^{0.75t} - 10$ and $y(t) = -10e^{0.75t} - 10$, respectively. The graphs these two functions, along with the graph of the solution $y(t) = 5e^{0.75t} - 10$, are illustrated in Panel (A) of Figure 10.2.3. Notice that the graph of the solution $y(t) = 5e^{0.75t} - 10$ shifts up shown by the upward pointing arrow if $C = 10$ and shifts down shown by the downward pointing arrow if $C = -10$. This shows that the graphs of the solutions will exhibit both scale effect and mirror effect depending upon the size and sign of the value of the constant C in their complementary functions.

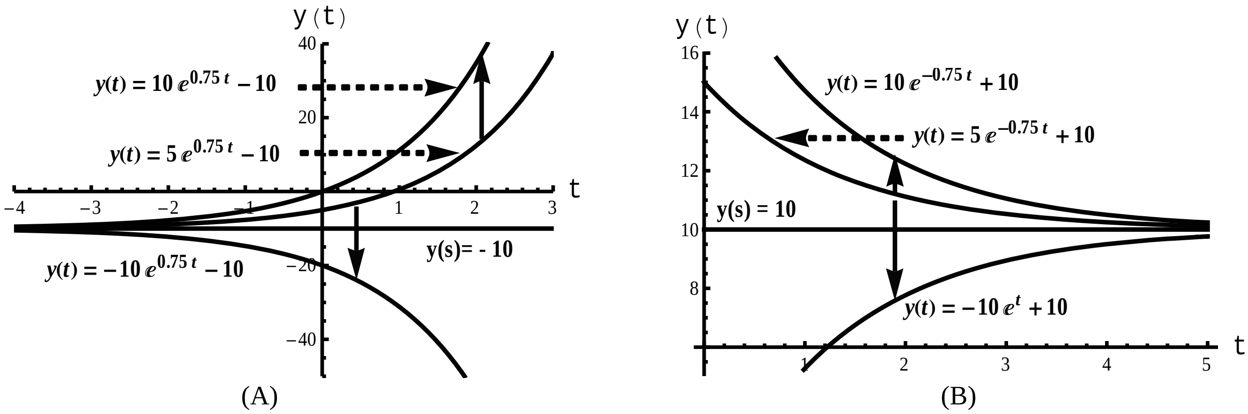


Figure 10.2.3

As another example, consider the solution $y(t) = 5e^{-0.75t} + 10$ to the differential equation $dy(t)/dt = -0.75y(t) + 7.5$ of the second example in present section. We illustrated the graph of this solution in Panel (B) of Figure 12.2.2 with that $C=5$. Suppose now that $C=10$ and $C=-10$. With these values, we obtain the solutions $y(t) = 10e^{-0.75t} + 10$ and $y(t) = -10e^{-0.75t} + 10$, respectively. The graphs of these two solutions, along with the graph of the solution $y(t) = 5e^{-0.75t} + 10$, are illustrated in Panel (B) of Figure 10.2.3. Notice that the graph of the solution shifts up shown by the upward pointing arrow if $C = 10$ and shifts down shown by the downward pointing arrow if $C = -10$. This shows that, as in the last example, the graphs of the solutions will exhibit both scale effect and mirror effect depending upon the size and sign of the value of the constant C in their complementary functions.

10.2.4 The phase diagrams

So far in the present chapter we were attempting to solve first-order, linear, autonomous, homogenous or non-homogenous differential equations explicitly. As in the case of difference equations, we were also interested in the behavior of the time paths of the solutions mainly to see whether they converge to, or diverge from, the long-run equilibrium or steady-state (horizontal axis when $b = 0$ or $-b/a$ when $b \neq 0$). For this we plotted the graphs of the solutions to few differential equations.

Again, as in the case of difference equations, there exists another geometric way to observe the long-run behavior of the solution path of a differential equation. This is based on phase diagram we introduced in Chapter 9. As we found there, this method is particularly useful when the equation is nonlinear or when there is a simultaneous system of differential equations, the topics which we will discuss later. Suppose that we have an equation similar to equation (10.2.1): $dy(t)/dt = ay(t) + b$. Since we have discussed the construction of phase diagrams in reasonable detail in Chapter 9, we do not repeat it here. Instead, we simply state that a phase diagram in the case of a differential equation is a graph that shows the plot of the differential equation, as the graph in either Panel (A), (C), or (E) in Figure 10.2.4. Notice that we do not make use of the 45° line in the present case.

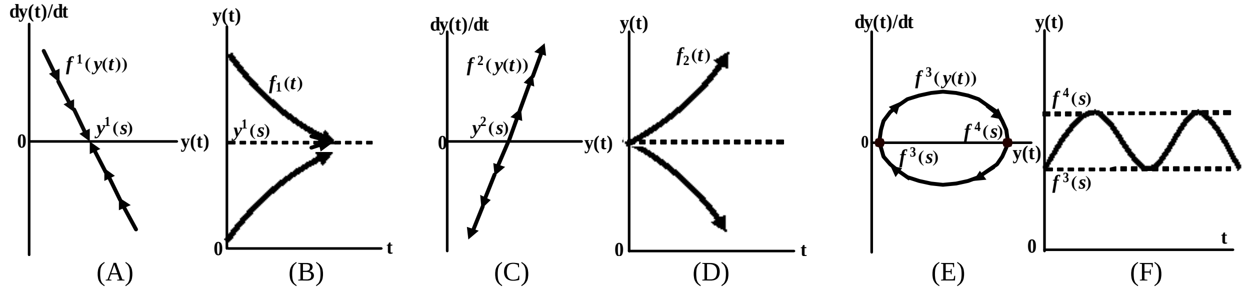


Figure 10.2.4

Suppose that we have three first-order, linear, autonomous difference equations: $dy(t)/dt = f^1[y(t)]$; $dy(t)/dt = f^2[y(t)]$, and $dy(t)/dt = f^3[y(t)]$. Suppose also that if we plot the derivative $dy(t)/dt$ of these equations as a function of $y(t)$ we obtain the phase lines represented by $f^1[y(t)], f^2[y(t)],$ *and* $f^3[y(t)]$ illustrated in Panels (A), (C), and (E), respectively, of Figure 10.2.4. Notice that $y(t)$ will remain stationary or will be in equilibrium or at steady-state when $dy(t)/dt = 0$. We can obtain this steady-state value of y, denoted by $y(s)$, by setting the differential equation to zero and solving for $y(s)$; that is by applying equation (10.2.10). These values are denoted by $y^1(s)$, $y^2(s)$ and $y^3(s)$ [or $y^4(s)$] in Panels (A), (C), and (E), respectively.

What are the curvatures of these phase lines? The answer depends on the sign of the derivative of the differential equation. If the derivative of $dy(t)/dt = ay(t) + b$ with respect to $y(t)$ is less than zero or if $a < 0$, then the phase line will fall from left to right (or rise from right to left) as shown by the phase line $f^1[y(t)]$ in Panel (A). Similarly, if that derivative is greater than zero, then the phase line will rise from left to right (or fall from right to left) as shown by the phase line $f^2[y(t)]$ in Panel (C). Notice that there are two steady-state values in Panel (E) and the phase line, which has infinite positive (negative) slope at the steady-state value $y^3(s)$ [$y^4(s)$], cycles between these two steady-state values.

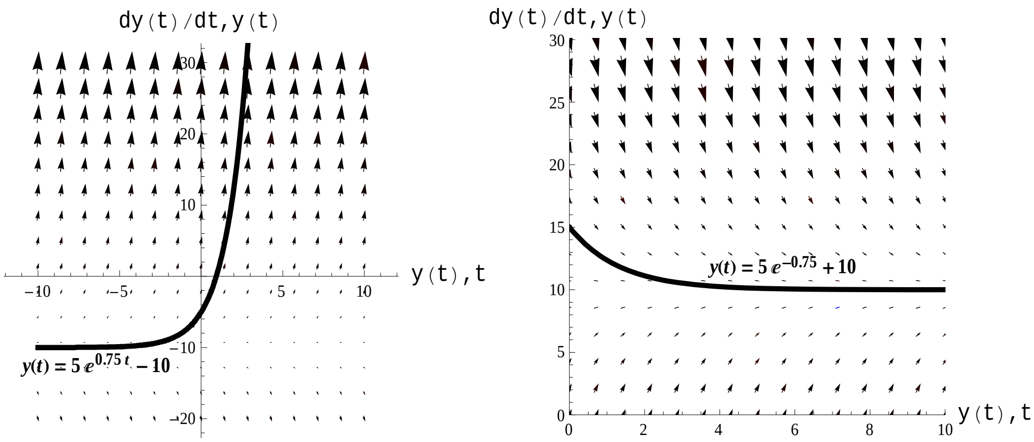
The reader must have noticed from Panel (A) that $dy(t)/dt$ decreases when $dy(t)/dt > 0$ or when $dy(t)/dt$ is above the horizontal axis; and $dy(t)/dt$ increases when $dy(t)/dt < 0$ or when $dy(t)/dt$ is below the horizontal axis. In other words, if $y(t) < y'(s)$, $y(t)$ increases; and if $y(t) > y'(s)$, $y(t)$ decreases. The opposite happens in Panel (C). Therefore, in Panel (A), $y(t)$ converges to the steady-state value $y^1(s)$ no matter what initial value we choose for $y(t)$. On the contrary, $y(t)$ diverges from the steady-state value $y^2(s)$ as shown in Panel (C). And in Panel (E) we see that $y(t)$ cycles between the two steady-state values without settling. These movements of $y(t)$ to, from, and around the steady-state value(s) are illustrated in Panels (B), (D) and (F) of Figure 10.2.4.

Our discussion above suggests that the slope of the phase line at its intersection point with the horizontal axis, such as $y^1(s)$, $y^2(s)$, and $y^3(s)$ or $y^4(s)$ in Panels (A), (C), and (E), respectively, determines whether the underlying variable of the differential equation converges to, or diverges from, or cycles between, the steady-state value(s). If the slope of the phase line is negative (positive) or infinite, then the variable will converge to (diverge from) or cycles between steady-state value(s), respectively. Notice that these results are exactly in accordance with our conclusions in the last section. Specifically, the graph of the solution $y(t) = 5e^{-0.75t} + 10$ to the equation $dy(t)/dt = -0.75y(t) + 7.5$ illustrated in Panel (B) of Figure 10.2.2 converged to the steady-state value 10 because the slope of the phase line was negative (that is, $a = -0.75$) in that example. Similarly, the graph of the solution $y(t) = 5e^{0.75t} - 10$ to the equation $dy(t)/dt = 0.75y(t) + 7.5$ illustrated in Panel (A) of Figure 10.2.2 diverged from the steady-state value -10 because the slope of the phase line was positive (that is, $a = 0.75$) in that example. Therefore, a simple check of the sign of the coefficient a, which is the same as the slope of the phase line, of the differential equation can help us judge whether the solution to the equation will converge to or diverge from (or cycles around) the steady-state value(s) even without solving the equation.

One can see in the literature that there exists an alternative method to draw inference about the convergence or the divergence of the solution to a differential equation without solving it. This alternative method utilizes the idea of **vector field** or **direction field**. A vector field is a space that contains a set of vectors. Notice that phase lines in Panels (A), (C) or (E) of Figure 10.2.4 are, in fact, vectors. We used only a few vectors corresponding to only few initial values of $y(t)$. If we choose many such values, we will obtain many corresponding vectors. The set of all these vectors is called a vector field. Therefore, the [$dy(t)/dt,\ y(t)$] space could be treated as a vector space. The direction of the vectors in the vector field can be used to infer about the convergence or divergence of the solution. If we have already solved the differential equation explicitly, then we can verify our inference if we plot the vector field and superimpose onto that the graph of the explicit solution.

As an example of the use of the plot of vector field in determining convergence without solving the differential equation, consider the first problem with equation $dy(t)/dt = 0.75y(t) + 7.5$ solved in the last section.

If we plot this equation choosing different values for $y(t)$, we obtain the vector field associated with this equation as illustrated in Panel (A) of Figure 10.2.5. Notice that all the vectors in this field move farther and farther from the steady-state value (-10) and tend to the north-east direction. This was the same conclusion we obtained from the graph of the solution to the differential equation presented in Panel (A) Figure 10.2.2. If we superimpose this graph onto the vector field, we see that our conclusion is verified. One can carry out a similar analysis with regard to the differential equation $dy(t)/dt = -0.75y(t) + 7.5$ in the second problem solved in the last section using its associated vector field and the graph of the equation's solution presented in Panel (B) of Figure 10.2.2. This analysis will yield the graph in Panel (B) of Figure 10.2.5.



(A) (B)

Figure 10.2.5

10.2.5 First-order, linear, non-autonomous differential equations

So far in the present chapter we were concerned with the solutions and their convergence of first-order, linear, autonomous differential equations. We are now ready to take up the topic of solutions to first-order, linear, non-autonomous differential equations. Notice that a first-order, linear, autonomous differential equation such as equation (10.2.1) becomes a first-order, linear, non-autonomous differential equation when either the coefficient a, or the term b, or both become some functions of time; that is, when we have either $a(t)$, or $b(t)$, or both $a(t)$ *and* $b(t)$. Therefore, we may write the general first-order, linear, non-autonomous differential equation as

$$dy(t)/dt = a(t)y(t) + b(t)\,. \qquad (10.2.13)$$

Notice that the only difference between equations (10.2.13) and (10.2.1) is that in the former the coefficient a and the term b are functions of time, while in the latter they are constants. How do we solve equations such as (10.2.13)? We cannot easily find the particular solution in the case of equation (10.2.13) as both the coefficient and the term are functions of time. However, we can still obtain the solutions to the equation and the procedure involved is outlined below.

Let us begin with the homogenous form $dy(t)/dt = a(t)y(t)$, treating $b(t) = 0$, of equation (10.2.13). If we follow exactly the same line of reasoning as that we followed in the derivation of equation (10.2.9), we can derive the complementary function in the present case as $y(c) = e^{\int a(t)dt - C}$. Let us now write, for convenience, the last equation as $y(c) = e^{\int a(t)dt - C} = e^{A(t)}$, where $A(t) = \int a(t)dt - C$ and C denotes the constant of integration.

We shall now multiply both sides of equation (10.2.13) by the term $e^{-A(t)}$, which is nothing but the reciprocal of the complementary function, to obtain $e^{-A(t)}[dy(t)/dt] = e^{-A(t)}a(t)y(t) + e^{-A(t)}b(t)$. The last equation can be rearranged to yield $e^{-A(t)}[dy(t)/dt] - e^{-A(t)}a(t)y(t) = e^{-A(t)}b(t)$, or $e^{-A(t)}[dy(t)/dt - a(t)y(t)] = e^{-A(t)}b(t)$. Notice that the L.H.S. of the transformed equation is the derivative of $e^{-A(t)}y(t)$ with respect to time. Therefore, the transformed equation can be written as $d[e^{-A(t)}y(t)]/dt = e^{-A(t)}b(t)$. We can now integrate both sides of the last equation to obtain $\int \{d[e^{-A(t)}y(t)]/dt\}dt = \int \left(e^{-A(t)}b(t)\right)dt$. We know that the integral of the derivative of a function is the function itself. Applying this result, we can write the L.H.S. of the last equation as $e^{-A(t)}y(t)$. This implies that we may rewrite the equation as $e^{-A(t)}y(t) = \int [e^{-A(t)}b(t)]dt + C$, which can be simplified to obtain

$$y(t) = e^{A(t)}\left(\int e^{-A(t)}b(t)dt + C\right) = Ce^{A(t)} + e^{A(t)}\int e^{-A(t)}b(t)dt, \qquad (10.2.14)$$

which gives the complete solution to equation (10.2.13). Notice that we obtained the solution in equation (10.2.14) to the equation (10.2.13) by multiplying the differential equation by a factor [in the present case, $e^{-A(t)} = e^{-\int a(t)dt - C}$], which made the differential equation amenable to direct integration. This factor is called an **integrating factor**.

So far in the present section we were concerned with determining the solution [in equation (10.2.14)] to equation (10.2.13) which contains both functional coefficient and term. Suppose now that, as a special case, we write equation (10.2.13) as

$$dy(t)/dt = a(t)y(t) + b, \qquad (10.2.15)$$

which is also a first-order, linear, non-autonomous differential equation. But, notice that in the last equation only the coefficient is a function of time while the term is a constant. How do we solve equations such as (10.2.15)? Since it is a special case of equation (10.2.13), we can deduce the solution to this equation from the solution in equation (10.2.14). Notice that the term b appears in this solution in its last part only and, therefore, only the last part of the equation will be affected. Since $b(t) = b$ in the equation, we can replace $b(t)$ in the last part of equation (10.2.14) with b and write it as

$$y(t) = e^{A(t)}\left(\int e^{-A(t)}bdt + C\right) = Ce^{A(t)} + e^{A(t)}\int e^{-A(t)}bdt = Ce^{A(t)} + e^{A(t)}b\int e^{-A(t)}dt, \qquad (10.2.16)$$

which is the solution to the differential equation (10.2.15). This suggests that the solution to equation (10.2.15), which is a special case of equation (10.2.13), in equation (10.2.16) is a special case of the solution in equation (10.2.14).

Suppose now that the term of the equation (10.2.13) is a function of time and that the coefficient is a constant. Then equation (10.2.13) can be written as

$$dy(t)/dt = ay(t) + b(t), \qquad (10.2.17)$$

which is also a first-order, linear, non-homogenous, non-autonomous differential equation. But, notice that in the last equation only the term is a function of time while the coefficient is a constant. How do we solve equations such as (10.2.17)? Since it is a special case of the equation (10.2.13), we can deduce, as before, the solution to this equation from the solution in equation (10.2.14). Since the coefficient is a constant, we can use equation (10.2.9) to find the complementary function as $y(c) = e^{at}$, where we have to avoid double counting. This means that the integrating factor when the coefficient is a constant is e^{at}. Therefore, we can replace the integrating factor $e^{-A(t)}$ in equation (10.2.14) with the integrating factor e^{at} to obtain

$$y(t) = e^{at}\left(\int e^{-at}b(t)dt + C\right) = Ce^{at} + e^{at}\int e^{-at}b(t)dt, \qquad (10.2.18)$$

which is the solution to the equation (10.2.17).

As the last case, suppose that both the coefficient and the term of equation (10.2.13) are constants. If the term b is constant, then the term $\int e^{-at}b(t)dt$ in equation (10.2.18) becomes $(-b/a)e^{-at}$. This implies that we can write equation (10.2.18) as $y(t) = Ce^{at} - e^{at}(b/a)e^{-at}$, or as $y(t) = Ce^{at} - b/a$, which is the same as the solution in equation (10.2.11) to equation (10.2.1). But, notice that equation (10.2.1) is a first-order, linear, non-homogenous, autonomous differential equation.

As an example, consider the differential equation $dy(t)/dt = 2ty(t) + 2t$. Notice that this is a first-order, linear, non-homogenous, non-autonomous differential equation and is identical with equation (10.2.13) for $a(t) = 2t$ *and* $b(t) = 2t$. Since both the coefficient and the term are functions of time, we can apply equation (10.2.14) to find the solution. But, to apply equation (10.2.14) we need to first find the integrating factor. Since $a(t) = 2t$, the integrating factor becomes $e^{-A(t)} = e^{-\int a(t)dt} = e^{-\int 2t\,dt} = e^{-t^2}$. We can now apply equation (10.2.14) to obtain the solution $y(t) = Ce^{A(t)} + e^{A(t)}\int e^{-A(t)}b(t)dt$, or $y(t) = Ce^{t^2} + e^{t^2}\int e^{-t^2}2tdt = Ce^{t^2} + 2e^{t^2}\int e^{-t^2}tdt$. The last equation can be further simplified to obtain $y(t) = Ce^{t^2} + 2e^{t^2}\int e^{-t^2}tdt = Ce^{t^2} + 2e^{t^2} \times (-e^{-t^2}/2) = Ce^{t^2} - 1$.

As another example, consider the equation $dy(t)/dt = 2ty(t) + 2$, which is identical with equation (10.2.15) for $a(t) = 2t$ *and* $b(t) = 2$. Since only the coefficient of the equation is a function of time, we can apply equation (10.2.16) to find its solution. Since there is no change in the coefficient of the equation, the integrating factor is, as in the last example, $e^{-A(t)} = e^{-\int a(t)dt} = e^{-\int 2t\,dt} = e^{-t^2}$. Therefore, applying equation (10.2.16), we can write the solution $y(t) = Ce^{A(t)} + e^{A(t)}b\int e^{-A(t)}dt = Ce^{t^2} + e^{t^2}b\int e^{-t^2}dt = Ce^{t^2} + 2e^{t^2}\int e^{-t^2}dt$.

As the last example, consider the equation $dy(t)/dt = 2y(t) + 2t$, which is identical with equation (10.2.17) for $a(t) = 2$ *and* $b(t) = 2t$. Since only the term of the equation is a function of time, we can apply equation (10.2.18) to find its solution. Therefore, applying equation (10.2.18), we obtain the solution $y(t) = Ce^{at} + e^{at}\int e^{-at}b(t)dt = Ce^{2t} + e^{2t}\int e^{-2t}2tdt = Ce^{2t} + 2e^{2t}\int e^{-2t}tdt$ $= Ce^{2t} + 2e^{2t}[(-e^{-2t}/4) - (-te^{-2t}/2)] = Ce^{2t} - t - (1/2)$.

10.2.6 First-order, nonlinear differential equations

So far in the present chapter we were concerned with the solutions, and their intertemporal behaviors, of first-order, linear, homogenous (autonomous or non-autonomous) differential equations. Let us now turn our attention to the first-order, nonlinear differential equations. As in the case of first-order, nonlinear, difference equations, obtaining explicit solutions to some nonlinear differential equations are often difficult. In this event, it is advisable to carry out a qualitative analysis of convergence of the underlying variable of the equation. This will be taken up in the present section. But, before this, we shall consider some forms of first-order, nonlinear differential equations that can be solved explicitly. There exist few specific methods to solve such equations depending upon their forms. These forms include **variable-separable equations**, **Bernoulli equations** and **linearizable equations**.

A first-order differential equation becomes nonlinear when the derivative, or the differential, or the variable of the equation (or both) appears with a power other than one and involves the cross products of the derivative or differential and the variable of the equation. Let us first consider the case of **variable-separation**. Assume that we have an equation of the form

$$f[y(t)][dy(t)/dt] = g(t). \tag{10.2.19}$$

Notice that in equation (10.2.19) f is a function of only $y(t)$ and g is function of only t. Equations such as (10.2.19) are said to be variable-separable equations because one can separate the $y(t)$ terms from the t terms in such equations. Then, one can apply the techniques of integration to find the solutions to such equations.

As an example, consider the differential equation $4y(t)[dy(t)/dt] = t$. Notice that this is similar to equation (10.2.6) if we assume that the term of the equation is zero. How do we solve differential equations such as these? In this equation $f[y(t)] = 4y(t)$ and $g(t) = t$ and, therefore, the equation is similar to equation (10.2.19). We can first convert the given equation into the form $f[y(t)]dy(t) = g(t)dt$ and, then, carry out integration on both sides of the equation. Then the L.H.S. becomes $\int 4y(t)dy(t) = 4[y(t)^2/2] + c_1 = 2y(t)^2 + c_1$ and the R.H.S.

becomes $\int t dt = (t^2/2) + c_2$. Equating these two integrals yields the solution: $2y(t)^2 + c_1 = (t^2/2) + c_2$ or, after simplification, $y(t) = [(t^2/4) + (C/2)]^{1/2}$, where $C = c_2 - c_1$.

We shall now consider Bernoulli equations. Suppose that we have a first-order, nonlinear, differential equation of the form

$$dy(t)/dt = a(t)y(t) + b(t)y(t)^n, \qquad (10.2.20)$$

where $n \neq 0,1$. An equation such as (10.2.20) is called a Bernoulli equation. Although Bernoulli equations are nonlinear, they can be converted into linear forms through suitable substitution. The conversion procedure is exposed below.

Suppose that we multiply both sides of equation (10.2.20) by $y(t)^{-n}$ to obtain $y(t)^{-n}[dy(t)/dt] = a(t)y(t)y(t)^{-n} + b(t)y(t)^n y(t)^{-n} = a(t)y(t)^{1-n} + b(t)$. Also suppose that we define a new variable $x(t) = y(t)^{1-n}$. From the last equation we obtain $dx(t)/dt = (1-n)y(t)^{-n}dy(t)/dt$, or $y(t)^{-n}dy(t)/dt = [1/(1-n)]dx(t)/dt$. Substituting the last result into $y(t)^{-n}dy(t)/dt = a(t)y(t)^{1-n} + b(t)$ we obtain $[1/(1-n)]dx(t)/dt = a(t)x(t) + b(t)$, or $dx(t)/dt = (1-n)a(t)x(t) + (1-n)b(t)$. Notice that the last equation is similar to equation (10.2.13), which can be solved by applying equation (10.2.14). Once we obtained the solution to the last equation in x, we can find $y(t)$ from $y(t) = x(t)^{1/(1-n)}$. Notice also that if the specific form of the last equation happens to be of the form of equations (10.2.15), (10.2.17), or (10.2.1), then we will have to apply equations (10.2.16), (10.2.18), or (10.2.11), respectively, to find the solution.

As an example, consider the equation $dy(t)/dt = 2y(t) + y(t)^3$, which is a Bernoulli equation and is of the form of equation (10.2.20) for $a(t) = 2, b(t) = 1,\ and\ n = 3$. Let us first multiply both sides of the given equation by $y(t)^{-3}$ to obtain $y(t)^{-3}dy(t)/dt = 2y(t)y(t)^{-3} + y(t)^3 y(t)^{-3} = 2y(t)^{-2} + 1$. The next step is to define the variable $x(t) = y(t)^{-2}$, which implies that $dx(t)/dt = -2y(t)^{-3}dy(t)/dt$, or $-(1/2)dx(t)/dt = y(t)^{-3}dy(t)/dt$. Substitution of this result into $y(t)^{-3}dy(t)/dt = 2y(t)^{-2} + 1$ yields $-(1/2)dx(t)/dt = 2x(t) + 1$, or $dx(t)/dt = -4x(t) - 2$. Notice that this differential equation is identical with equation (10.2.1) for $a = -4\ and\ b = -2$ having the solution in equation (10.2.11): $x(t) = Ce^{at} - b/a = Ce^{-4t} - (-2/-4) = Ce^{-4t} - 1/2$. Since $y(t) = x(t)^{-1/2}$, the last equation can be converted into $y(t)$ and the result will be $y(t) = [Ce^{-4t} - 1/2]^{-1/2}$. Therefore, the solution to the differential equation $dy(t)/dt = 2y(t) + y(t)^3$ is $y(t) = [Ce^{-4t} - 1/2]^{-1/2}$.

We observed in the present section that if the given differential equation happens to be a variable-separable or a Bernoulli equation, then we could find the explicit solution to the equation using the methods exposed so far. Unfortunately, we are not often that lucky to come across such equations in the subjects of our interest. One may frequently come across equations that are neither variable-separable nor Bernoulli equations. How do we, then, solve such differential equations? One way ahead is to linearize nonlinear differential equations using Taylor series introduced in Section 3.6. The procedure of linearizing a first-order, nonlinear differential equation is similar to linearizing a first-order, nonlinear difference equation exposed in Section 9.2.9.

Suppose that our general first-order differential equation is given by the function

$$dy(t)/dt = f[y(t)], \qquad (10.2.21)$$

where $f[y(t)]$ is assumed to be continuously differentiable and nonlinear. Notice that we have not given specific form to equation (10.2.21) and suppose that it cannot be solved explicitly by any of the methods we exposed so far in the present chapter. Then, as mentioned above, one way ahead is to linearize the equation (10.2.21) using Taylor series. Suppose that this equation possesses a steady-state value at $y(s)$. Now taking a first-order Taylor expansion around the steady-state value $y(s)$ yields $f[y(t)] = f[y(s)] + f'[y(s)][y(t) - y(s)]$, where we have omitted the remainder term for convenience. Since $dy(t)/dt = f[y(t)] = 0$ at the steady-state, we have $f[y(s)] = 0$. Therefore, the above Taylor expansion reduces to a relatively simple equation $f[y(t)] = f'[y(s)][y(t) - y(s)]$. Thus, we can write the linearized form of the differential equation as $dy(t)/dt = f[y(t)] = f'[y(s)][y(t) - y(s)]$, which can be written, for convenience, as $dy(t)/dt = f'[y(s)]y(t) - f'[y(s)]y(s)$, which is now similar to equation (10.2.1), for $a = f'[y(s)]$, and $b = -f'[y(s)]y(s)$ having the solution in equation (10.2.11). Applying equation (10.2.10) we can obtain the particular solution $y(s) = p = -b/a = -\{-f'[y(s)]y(s)/f'[y(s)]\}$. The complementary function can be found, by applying equation (10.2.9), as $y(c) = Ce^{at}$, where $C = y(t) - y(s) = y(0) - y(s)$. We can now combine the particular solution and the complementary function by applying equation (10.2.11) to obtain the complete solution

$$y(t) = Ce^{at} + y(s) = \left(y(0) - y(s)\right)e^{f'(y(s))t} + y(s). \tag{10.2.22}$$

So far in the present section we were concerned with the solution of first-order, nonlinear differential equations. We found above that if the equations were of the forms of variable-separable, Bernoulli, or linearizable equations, then we could solve the equations and determine whether the solutions converged to or diverged from the steady-state values. However, many of the equations we often find in the subjects of our interest do not possess the above mentioned forms and, therefore, deriving explicit solutions to those equations are often difficult. In such situations, as we mentioned in Section 10.2.4, qualitative analyses of the intertemporal behavior of the underlying variables of the differential equations could be carried out through the application of phase diagrams. It must be emphasized that phase diagrams are applicable not just to linear differential equations; they are equally applicable to their nonlinear forms. We have already introduced phase diagrams of linear differential equations in Section 10.2.4. Therefore, we will not repeat the construction of phase diagrams here and, instead, we shall apply them to few specific examples.

As an example, consider equation $dy(t)/dt = f[y(t)] = y(t) - 2y(t)^2$, which is of the form of Bernoulli equation (10.2.20) for $a(t) = 1, b(t) = -2, \text{ and } n = 2$. Therefore, we can obtain the solution to it following the method we discussed immediately after equation (10.2.20). Suppose now that we do not know the solution of Bernoulli equations and that linearization of the equation through Taylor series is a cumbersome process. Under these circumstances, an alternative is to draw qualitative inference about the variable $y(t)$ through the phase diagram. The phase diagram for the present differential equation is illustrated in Panel (A) of Figure 10.2.6.

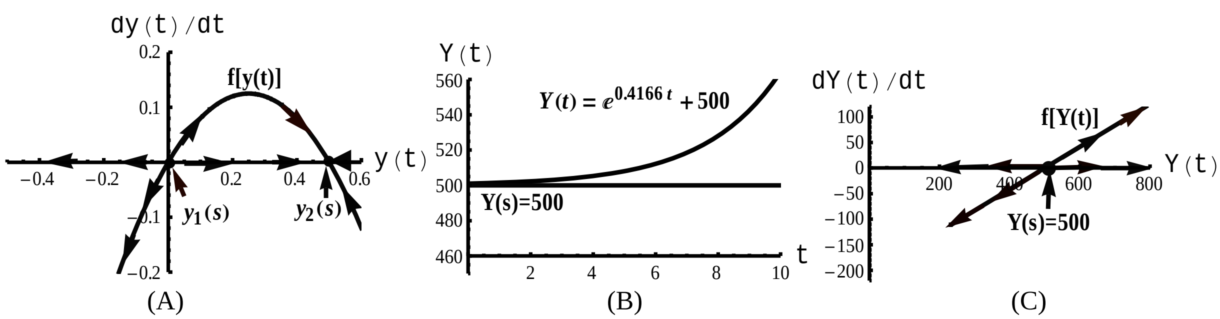


Figure 10.2.6

We can now discuss the qualitative nature of the solution to equation $dy(t)/dt = f[y(t)] = y(t) - 2y(t)^2$ illustrated in the phase diagram in Panel (A) of Figure 10.2.6. Let us first determine the steady-state value or the particular solution. We know that the steady-state value of the variable $y(t)$ occurs when $dy(t)/dt = 0$. Therefore, setting the equation to zero and simplifying yields two steady-state values: $y_1(s) = 0$ and $y_2(s) = 0.5$. We stated in Section 10.2.4 that the variable $y(t)$ would converge to (diverge from) the steady-state value if the slope of the phase line at its intersection point with the horizontal axis was negative (positive). This suggests that we need to find the slope of the phase line at the two steady-states. The slope of the phase line at any point on it is given by the first derivative of the differential equation: $f'[y(t)] = 1 - 4y(t)$, which suggests that the slope of the phase line at the first steady-state is $f'[y_1(s)] = 1 - 4y_1(s)$, or $f'(0) = 1 - 4 \times 0 = 1$. The slope of the phase line at the second steady-state is $f'[y_2(s)] = 1 - 4y_2(s)$ of $f'(0.5) = 1 - 4 \times 0.5 = -1$.

Notice that the slope of the phase line at $y_1(s) = 0$ is 1 and that at $y_2(s) = 0.5$ is -1. The phase line in Panel (A) of Figure 10.2.6 shows that if $dy(t)/dt > 0$ or if $dy(t)/dt$ is above the horizontal axis, $y(t)$ increases to $y_2(s) = 0.5$; and if $dy(t)/dt < 0$ or if $dy(t)/dt$ is below the horizontal axis, $y(t)$ decreases to $y_2(s) = 0.5$. These are shown by the $y_2(s) = 0.5$-pointing arrows on both the horizontal axis and the phase line in the panel. This means that no matter what value (above zero) we choose as the initial value for $y(t)$, $y(t)$ converges to $y_2(s) = 0.5$ and, therefore, $y_2(s) = 0.5$ is a stable equilibrium point. But, the opposite happens at the first steady-state point $y_1(s) = 0$. If $y(t)$ increases from this steady-state, then $y(t)$ keeps on increasing; and if $y(t)$ decreases, then $y(t)$ keeps on decreasing. These are shown by the $y_2(s) = 0$-departing arrows on both the horizontal axis and the phase line. This means that no matter what value we choose as the initial value for $y(t)$, $y(t)$ diverges from $y_1(s) = 0$ and, therefore, $y_1(s) = 0$ is an unstable equilibrium point.

The reasons why $y_1(s) = 0$ and $y_2(s) = 0.5$ are unstable and stable equilibrium points respectively are the facts that the slope of the phase line at $y_1(s) = 0$ is positive (1) and the slope of the phase line at $y_2(s) = 0.5$ is negative (-1). These results are perfectly in accordance with our conclusion in Section 10.2.4. The reader must have noticed that we obtained the above information on the intertemporal behavior of the variable of the differential equation, even without solving the equation explicitly, through a qualitative analysis using a phase diagram. Had we solved the equation explicitly, then we could have also plotted the graph of the solution along side the panel in Figure 10.2.6 [as Panels (B), (D), and (F) in Figure 10.2.4]. We could have also obtained information similar to the one we obtained through the phase diagram above had we constructed a vector field and plotted the explicit solution in the vector field as in Panels (A) and (B) of Figure 10.2.5.

10.2.7 Application examples

Example 1: Consider the first example in Section 10.2.10. In a continuous time context, suppose that the aggregate consumption at time t in an economy is given by $C(t) = C(0) + cY(t)$, where C_0, c and $Y(t)$ denote autonomous consumption, the marginal propensity to consume, and the national income, respectively, at time t. Also suppose that the national income at time t in the economy is given by $Y(t) = C(t) + I(t)$, where $I(t)$ denotes total investment in the economy at time t, which is assumed to be a function of the rate of change of aggregate consumption: $I(t) = m(dC/dt)$, where $m > 0$. Set up the differential equation in $Y(t)$, derive the time path of the economy's national income, and determine whether this path converges to or diverges from the steady state, if it exists, using a phase diagram if $C(0) = 100, Y(0) = 501, c = 0.8,$ *and* $m = 0.6$. All values are in billions of dollars.

Solution: This is a **continuous time income determination** problem. Let us first set up the required differential equation. We are given that $Y(t) = C(t) + I(t)$ into which we substitute $C(t) = C(0) + cY(t)$ and $I(t) = m(dC/dt)$ to obtain $Y(t) = C(0) + cY(t) + m[dC(t)/dt]$. Notice that the **consumption function** $C(t) = C(0) + cY(t)$ implies that $dC(t)/dt = cdY(t)/dt$, which can be substituted into the last result to yield $Y(t) = C(0) + cY(t) + m[dC(t)/dt] = C(0) + cY(t) + c[dY(t)/dt]$. The last equation can be rearranged to obtain $dY(t)/dt = [(1-c)/mc]Y(t) - [C(0)/mc]$, which is the required differential equation and is similar to equation (10.2.1) for $a = [(1-c)/mc]$ *and* $b = -[C(0)/mc]$, with the solution in equation (10.2.11). To find the complementary function we can apply equation (10.2.9): $Y(c) = Ce^{at}$, where C denotes the constant of integration. Using $a = [(1-c)/mc]$, we can rewrite the complementary function as $Y(c) = Ce^{[(1-c)/mc]t}$. Applying equation (10.2.10) the particular solution becomes $Y(s) = -b/a = -\{[-C(0)/mc]/[(1-c)/mc]\} = C(0)/(1-c)$. Lastly, we shall combine the complementary function with the particular solution to obtain the complete solution: $Y(t) = Y(c) + Y(s) = Ce^{[(1-c)/mc]t} + C(0)/(1-c)$. Let us now substitute the values $c = 0.8, m = 0.6$ *and* $C(0) = 100$ into the complete solution to obtain $Y(t) = Ce^{[(1-c)/mc]t} + C(0)/(1-c)$ $= Ce^{0.4166\ t} + 500$. We shall now definitize the complete solution to obtain the complete and definite solution using the initial condition $Y(0) = 501$. Using this initial condition we find that $C = 1$. Therefore, the complete and definite solution is $Y(t) = e^{0.4166t} + 500$, the graph of which is illustrated in Panel (B) of Figure 10.2.6.

Let us now draw the phase diagram for the differential equation $dY(t)/dt = f[Y(t)] = 0.4166Y(t) - 208.33$. Following the same procedure as that we used in Sections 10.2.4 and 10.2.6, we can draw the phase diagram as illustrated in Panel (C) of Figure 10.2.6, from which see that $Y(t)$ increases (decreases) when $dY(t)/dt > 0$ [$dY(t)/dt < 0$]. This implies that $Y(t)$ diverges from its steady-state value $Y(s) = 500$, which is obtained by equating the differential equation to zero and solving for $Y(t = s)$. Notice that the phase line is a straight line and, therefore, its slope (at any point on it, including at the steady-state point) is the constant value 0.4166 > 0. Since the slope of the phase line at the steady-state is positive, as per the conditions stated in Sections 10.2.4 and 10.2.6, $Y(t)$ must diverge from the steady-state. This is what happens in the phase diagram in Panel (C) of Figure 10.2.6. Notice that a similar conclusion can be inferred from the graph of the solution to the differential equation illustrated in Panel (B) of the same figure.

Example 2: Consider the problem in the cobweb model of price determination in example 2 of Section 9.2.10. In a continuous time framework, suppose that the demand for a good at time t is given by the function $q_d(t) = m - np(t)$, where $q_d(t)$ and $p(t)$ denote the quantity demanded and the price of the good, respectively, at time t, $m > 0$ *and* $n > 0$; and that the supply of the good is given by the function $q_s(t) = u + vp(t)$, where $q_s(t)$ denotes the quantity supplied of the good at time t, $u > 0$ *and* $v > 0$, respectively. Also suppose that the rate of change of price is given by the function $dp(t)/dt = k[q_d(t) - q_s(t)]$, where k is a constant. Set up the differential equation in price and solve it assuming that $m = 15, n = 2, u = 5, v = 1, k = 0.5,$ *and* $p(0) = 5.33$. Does the path of price $p(t)$ converge to the equilibrium level of price $p(s)$ (if it exists) and, if so, what is the equilibrium price?

Solution: This is an example of a version of **Walrasian price-adjustment model**. Substituting $q_d(t) = m - np(t)$ and $q_s(t) = u + vp(t)$ into $dp(t)/dt = k[q_d(t) - q_s(t)]$ yield the required differential equation: $dp(t)/dt = k[q_d(t) - q_s(t)] = k[m - np(t) - u - vp(t)] = km - knp(t) - ku - kvp(t)$, which can be simplified as $dp(t)/dt = -(kn + kv)p(t) + (km - ku)$ and is similar to equation (10.2.1) for $a = -(kn + kv) = -k(n + v)$ and $b = (km - ku) = k(m - u)$ having the solution in equation (10.2.11). The complementary function can be found by applying equation (10.2.9): $p(c) = Ce^{at}$. Substituting $a = -k(n + v)$, the complementary function can be written as $p(c) = Ce^{at} = Ce^{-k(n+v)t}$, where C denotes the constant of integration. Applying equation (10.2.10), the particular solution becomes $p(s) = -b/a = -[k(m-u)]/[-k(n+v)] = [(m-u)/(n+v)]$. Lastly, we shall combine the complementary function with the particular solution to obtain the complete solution: $p(t) = p(c) + p(s) = Ce^{-k[n+v]t} + [(m-u)/(n+v)]$. Let us now substitute the values $m = 15, n = 2, u = 5, v = 1,$ *and* $k = 0.5$ into the complete solution to obtain $p(t) = Ce^{-k[n+v]t} + [(m-u)/(n+v)] = Ce^{-0.5[2+1]t} + [(15-5)/(2+1)] = Ce^{-1.5t} + 3.33$. We definitize the last result to obtain the complete and definite solution using the initial condition $p(0) = 5.33$. Using this initial condition we find that $C = 2$. Therefore, the complete and definite solution becomes $p(t) == 2e^{-1.5t} + 3.33$, the graph of which is illustrated in Panel (A) of Figure 10.2.7. The phase diagram of the differential equation $dp(t)/dt = f[p(t)] = -1.5p(t) + 5$ is shown in Panel (B) of the same figure. Since $f'[p(t)] = -1.5 < 0$, as can be seen from Panel (B), price $p(t)$ converges to the steady-state or equilibrium price $p(s) = 3.33$.

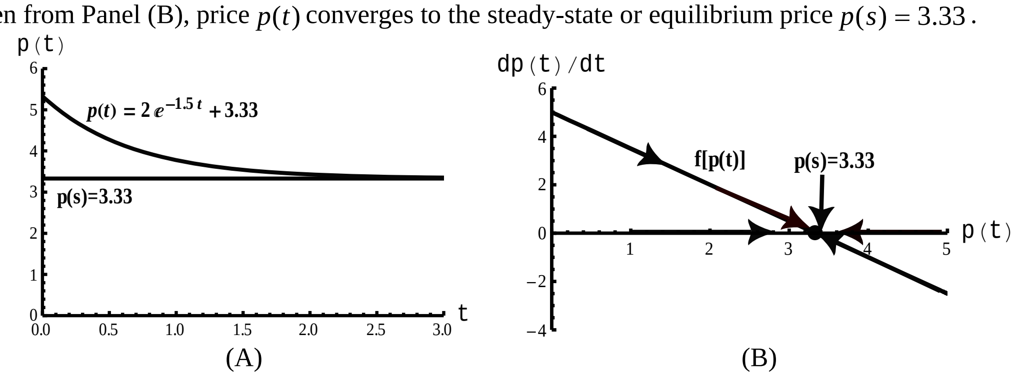


Figure 10.2.7

Example 3: Suppose that the rate of change of a sum of money is given by the function $dF(t)/dt = rF(t)$, where $F(t)$ *and* r denote the sum of money at time t and rate of interest compounded continuously, respectively. Determine the sum of money as a function of time and show the intertemporal behavior of the sum of money with a phase diagram if $r = 0.10$ *and* $F(0) = 100$.

Solution: Notice that the equation that $dF(t)/dt = r.F(t)$ is a homogenous differential equation and that it can be rewritten as $1/F(t)[dF(t)/dt] = r$ or as $1/F(t)[dF(t)] = rdt$, which is now in the form of variable-separable equation (10.2.19). Therefore, we can perform direct integration of the equation to find the time path of the sum of money. Integration of both sides of the last equation yields $\int [1/F(t)]dF(t) = \int rdt = r\int dt$; or $\ln F(t) + c_1 = rt + c_2$; or $\ln F(t) = rt + c_3$, where $c_3 = c_2 - c_1$. The last result is equivalent to $e^{\ln F(t)} = e^{rt+c_3}$; or to $F(t) = e^{c_3}e^{rt}$; or to $F(t) = Ce^{rt} = Ce^{0.1t}$, where $C = e^{c_3}$ *and* $r = 0.1$. The equation $F(t) = Ce^{0.1t}$ gives the solution and is the required function. Let us now definitize the solution using the initial condition $F(0) = 100$ to obtain the definite solution $F(t) = 100e^{0.1t}$, the graph of which is illustrated in Panel (A) of Figure 10.2.8. The phase diagram of the differential equation $dF(t)/dt = f[F(t)] = rF(t) = 0.10 \times F(t)$ is illustrated in Panel (B) of the same figure. Since $f'[F(t)] = 0.1 > 0$, as can be seen from Panel (B), $F(t)$ diverges from the steady-stage value $F(s) = 0$.

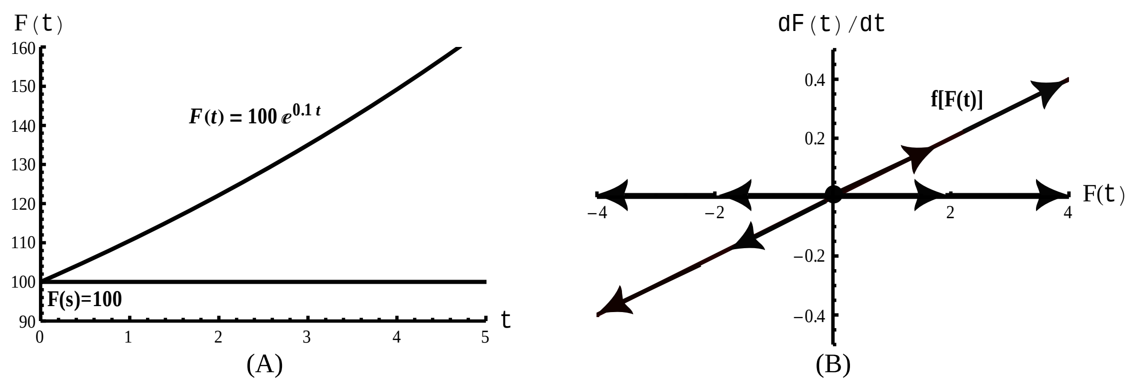
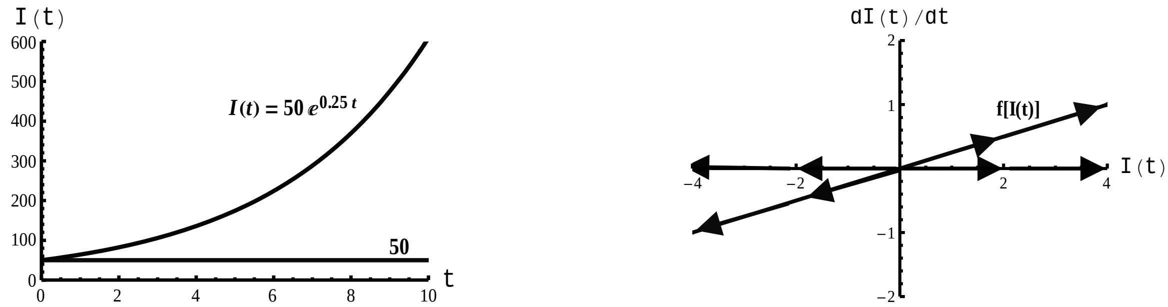


Figure 10.2.8

Example 4: Suppose that the rate of change of **aggregate demand** in an economy at time t is given by the function $dY(t)/dt = (1/s)[dI(t)/dt]$, where $Y, s,$ *and* I denote national income, MPS, and investment, respectively, and MPS is assumed to be constant and lies between 0 and 1. Also suppose that the rate of change of **aggregate supply** in the economy is given by the function $dQ(t)/dt = (1/c)I(t)$, where Q *and* c denote total supply of goods and services and **marginal input-output ratio**, respectively. Determine the rate at which invest must grow while maintaining **full-employment** in the economy if $s = 0.2, c = 0.8$ *and* $I(0) = 50$. All values are in billions of dollars.

Solution: This is an example of **Domar's model** of economic growth. Since full-employment is assumed, let us equate aggregate demand and supply to obtain $(1/s)[dI(t)/dt] = (1/c)I(t)$, which can be rearranged to yield $[1/I(t)]dI(t) = (s/c)dt$, which is in the form of variable-separable equation (10.2.19). Therefore, as in the last example, we shall integrate both sides of the last equation to get $\int [1/I(t)]dI(t) = \int (s/c)dt = (s/c)\int dt$; or $\ln I(t) + c_1 = (s/c)t + c_2$; or $\ln I(t) = (s/c)t + c_3$, where $c_3 = c_2 - c_1$. The last result is equivalent to $e^{\ln I(t)} = e^{(s/c)t+c_3}$; or $I(t) = e^{c_3}e^{(s/c)t}$; or $I(t) = Ce^{(s/c)t} = Ce^{0.25t}$, where $C = e^{c_3}$ *and* $s/c = 0.2/0.8 = 0.25$. The equation $I(t) = Ce^{0.25t}$ represents the solution in the present example and, therefore, it shows that investment must grow at the rate of 25 percent per year to maintain full-employment in the economy. Definitizing the solution with $I(0) = 50$ yields $I(t) = 50e^{0.25t}$, the graph of which is illustrated in Panel (A) of Figure 10.2.9. The phase diagram of the differential equation $dI(t)/dt = f[I(t)] = (s/c) \times I(t) = 0.25 \times I(t)$ is illustrated in Panel (B) of the same figure. Since $f'[I(t)] = 0.25 > 0$, as can be seen from the Panel (B), $I(t)$ diverges from the steady-state value $I(s) = 0$.

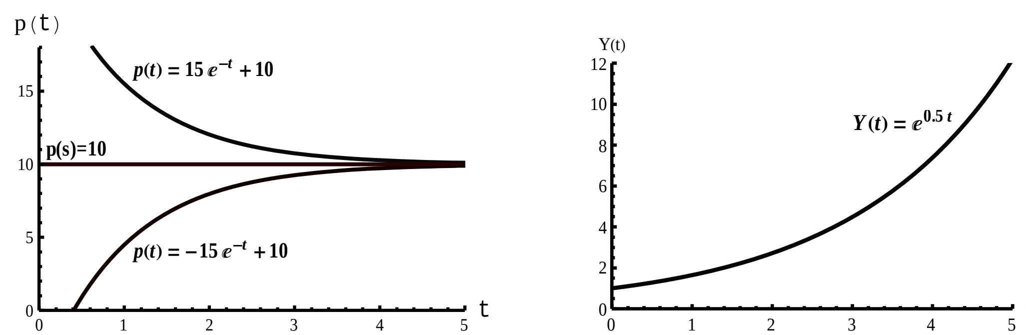


(A) (B)

Figure 10.2.9

Example 5: Assume that the quantities demanded and supplied of a good at time t are given by $q_d(t) = d_0 - d_1 p(t) + d_2 dp(t)/dt$ and $q_s(t) = s_0 + s_1 p(t) + s_2 dp(t)/dt$, respectively; where $q_d(t), q_s(t), \; and \; p(t)$ denote the quantities demanded and supplied, and the price of the good, respectively. We also assume that $d_0, d_1, d_2, s_0, s_1, \; and \; s_2$ are constants and that $s_2 - d_2 = s_1 + d_1$. Determine the time path of price assuming that $d_0 = 100, d_1 = 3, d_2 = 3, s_0 = -50, s_1 = 2, \; and \; s_2 = 8$, and that $p(0) = 15$ or $p(0) = 7$.

Solution: Let us first equate the demand and supply functions and manipulate the result to obtain $[-(s_2 - d_2)/\{(s_1 + d_1)p(t) - (s_0 + d_0)\}]dp(t) = dt$, which is in the form of variable-separable equation (10.2.19). Therefore, integration of both sides of the last equation yields $\int [(s_2 - d_2)/\{(s_1 + d_1)p(t) - (s_0 + d_0)\}]dp(t) = -\int dt$; or $\ln[(s_1 + d_1)p(t) - (s_0 + d_0)] + c_1 \;\; = -t - c_2$; or $\ln[(s_1 + d_1)p(t) - (s_0 + d_0)] = -t + c$, where $c = c_2 - c_1$, which can be simplified to yield $[(s_1 + d_1)p(t) - (s_0 + d_0)] = e^c e^{-t} = Ce^{-t}$, where $C = e^c$; or $p(t) = [\{Ce^{-t}/(s_1 + d_1)\} + (s_0 + d_0)/(s_1 + d_1)]$. Substitution of the values $d_0 = 100, d_1 = 3, d_2 = 3, s_0 = -50, s_1 = 2, \; and \; s_2 = 8$ into the last result yields $p(t) = [Ce^{-t}/(2+3)] + [(-50+100)/(2+3)] = [Ce^{-t}/5] + 50/5 = [Ce^{-t}/5] + 10$. Definitizing this solution with $p(0) = 15$ and $p(0) = 7$ yields the definite solutions $p(t) = 15e^{-t} + 10$ with $C = 15$ and as $p(t) = -15e^{-t} + 10$ with $C = -15$, the graphs of which are illustrated in Panel (A) of Figure 10.2.10.



(A) (B)

Figure 10.2.10

Example 6: Assume that the aggregate saving at time t in an economy is given by the function $S(t) = mY(t)$, where $S, m > 0, \; and \; Y$ denote aggregate saving, constant MPS, and national income, respectively. Also suppose that the investment in the economy is a function of the rate of change of the economy's national income: $I(t) = n[dY(t)/dt]$, where $I(t)$ denotes aggregate investment in the economy and n is a constant greater than zero. Determine the time path of national income assuming full-employment and $m = 0.2, n = 0.4, \; and \; Y(0) = 1$.

Solution: This is an example of the ***continuous time version*** of Harrod's model of economic growth. Notice that at full employment we have $S(t) = I(t)$, or $mY(t) = n[dY(t)/dt]$, which can be written as $dY(t)/Y(t) = (m/n)dt$, or $[1/Y(t)]dY(t) = (m/n)dt$, which is in the form of separable-variable equation (10.2.9).

Therefore, integration of both sides of the last equation yields $\int[1/Y(t)]dY(t) = \int(m/n)dt$; or $\ln Y(t) + c_1 = (m/n)t + c_2$; or $\ln Y(t) = (m/n)t + c_2 - c_1$; or $\ln Y(t) = (m/n)t + c$, where $c = c_2 - c_1$. The last equation can be written as $e^{\ln Y(t)} = e^c e^{(m/n)t}$ or as $Y(t) = Ce^{(m/n)t}$, where $C = e^c$. Using the given values $m = 0.2$ *and* $n = 0.4$, the last result can be written as $Y(t) = Ce^{(0.2/0.4)t} = Ce^{0.5t}$. Definitizing this result using the initial condition $Y(0) = 1$ we obtain $Y(t) = e^{0.5t}$, which gives the time path of the economy's national income and the graph which is illustrated in Panel (B) of Figure 10.2.10.

Example 7: Suppose that the total output $[Y(t)]$ at time t in an economy is produced, using $K(t)$ units of capital, $L(t)$ units of labor, and technology $A(t)$, given by the Cobb-Douglas production function $Y(t) = F[K(t), A(t)L(t)] = K(t)^\alpha (A(t)L(t))^{1-\alpha}$, where $0 < \alpha < 1$. Then output per unit of effective labor is given by $y(t) = f[k(t)] = Y(t)/A(t)L(t) = K(t)^\alpha/[A(t)L(t)]^\alpha = k(t)^\alpha$ and capital per unit of effective labor is given by $k(t) = K(t)/A(t)L(t)$. Also suppose that the rate of change of capital is given by $dK(t)/dt = sY(t)$, where $0 < s < 1$ denotes the constant saving rate in the economy, and that labor and technology grow at positive but constant rates n *and* g, respectively. Derive the differential equation in $k(t)$. What conclusion can be drawn about the intertemporal behavior of $k(t)$?

Solution: This is an example of the **continuous time version** of Solow model of economic growth. Let us first derive the differential equation in $k(t)$. For this we differentiate $k(t) = K(t)/A(t)L(t)$ with respect to time to yields $dk(t)/dt = \{A(t)L(t)\dot{K}(t) - K(t)[\dot{A}(t)L(t) + A(t)\dot{L}(t)]\}/[A(t)L(t)]^2$, which can be simplified to obtain $dk(t)/dt = [\dot{K}(t)/A(t)L(t)] - [K(t)/A(t)L(t)]\{[\dot{A}(t)L(t)/A(t)L(t)] + [\dot{L}(t)A(t)/A(t)L(t)]\}$. Using the given definitions, the last result can be written as $dk(t)/dt = [sY(t)/A(t)L(t)] - k(t)[g+n] = sk(t)^\alpha - (g+n)k(t)$. Therefore, the required differential equation is $dk(t)/dt = sk(t)^\alpha - (g+n)k(t)$.

Notice that the derived equation is a first-order, nonlinear differential equation similar to Bernoulli equation (10.2.20). One can now use either the method of solving Bernoulli equations or the Taylor series to linearize it and find an approximate solution. But these methods involve tedious algebraic manipulations and, therefore, we do not follow them. Instead, as we carried out a qualitative analysis of the discrete time version of the same Solow model in example 7 of Section 9.2.10, we shall carry out an analysis of the present continuous time version of the model using phase diagram.

Notice also that the equation contains two terms on its RHS: $sk(t)^\alpha$ *and* $(g+n)k(t)$. The former and the latter are nonlinear and linear terms, respectively. When we plot these two terms (functions) with $\alpha = 0.5, s = 0.25,$ *and* $g = n = 0.04$, we obtain the graph in Panel (A) of Figure 10.2.11. The graph of $sk(t)^\alpha$ increases at a diminishing rate and the graph of the term $(g+n)k(t)$ is a straight line passing through the origin. These imply that the graph of $sk(t)^\alpha$ will cross the graph of $(g+n)k(t)$ at two points: at (0,0) and at $sk(t)^\alpha = (g+n)k(t)$. The latter point is S corresponding to $k(s) = [(g+n)/s]^{1/(\alpha-1)}$ in Panel (A) of the figure.

Let us now plot the phase line of the differential equation $dk(t)/dt = h[k(t)] = sk(t)^\alpha - (g+n)k(t)$. But, before this, we need to specify its curvature. The first derivative of the phase line is $h'[k(t)] = s\alpha k(t)^{\alpha-1} - (g+n)$. As long as $\alpha k(t)^{\alpha-1} > (g+n)$, we have $h'[k(t)] = s\alpha k(t)^{\alpha-1} - (g+n) > 0$. The second derivative of the phase line is $h''[k(t)] = s\alpha(\alpha-1)k(t)^{\alpha-2} < 0$ (because $0 < \alpha < 1,$ *and* $(\alpha - 1) < 0$). These two derivatives point to the fact that the phase line is strictly concave. Moreover, we know that $k(t)$ will be at steady-state when $dk(t)/dt = 0$, which implies that $dk(t)/dt = h(k(t)) = sk(t)^\alpha - (g+n)k(t) = 0$, or $sk(t)^\alpha - (g+n)k(t) = 0$, or $k(s) = [(g+n)/s]^{1/(\alpha-1)}$. We can now evaluate the above first derivative at the steady-state value, which gives $h'[k(s)] = s\alpha\{[(g+n)/s]^{1/(\alpha-1)}\}^{\alpha-1} - (g+n) = \alpha(g+n) - (g+n) = (\alpha-1)(g+n)$. Since $0 < \alpha < 1, (\alpha-1) < 0$ and $(g+n) > 0$, $(\alpha-1)(g+n) < 0$ or $h'[k(s)] < 0$. Therefore, using our results in Section 10.2.6, we can deduce that k converges to the steady-state level $k(s) = [(g+n)/s]^{1/(\alpha-1)}$. This is shown by the arrows in Panel (B) of

Figure 10.2.11. Notice that the gap between the phase line and the horizontal axis in Panel (B) is equal to the gap between the curves of $sk(t)^{\alpha}$ *and* $(g+n)k(t)$ in Panel (A).

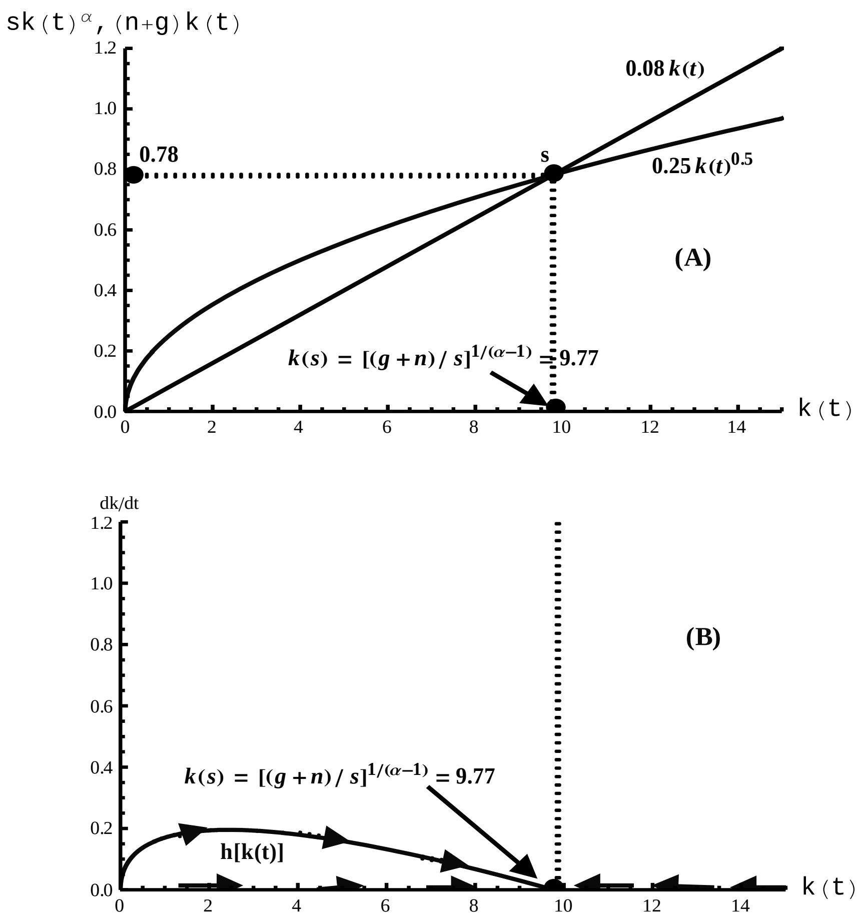


Figure 10.2.11

10.2.8 Exercises

1. Solve the following differential equations. Definitize the solutions using $y(0)=1$ when $t=0$ and verify your answers.

(i) $dy(t)/dt-4y(t)=0$ (ii) $dy(t)/dt-4-4y(t)=4$ (iii) $dy(t)/dt+4y(t)=0$
(iv) $dy(t)/dt=y(t)$ (v) $dy(t)/dt=2-y(t)$ (vi) $dy(t)/dt-2y(t)=2-y(t)$.

2. Sketch the graphs of the solutions to the differential equations in exercise 1 above. Draw the phase diagrams of the differential equations in exercise 1 above and determine whether $y(t)$ converges to or diverges from steady-states (if any).

3. Solve the following differential equations. Definitize the solutions using $y(0)=1$ when $t=0$.

(i) $dy(t)/dt=ty(t)+2t$ (ii) $dy(t)/dt=ty(t)+2t^2$ (iii) $dy(t)/dt=ty(t)+2$
(iv) $dy(t)/dt=t^2y(t)+2$ (v) $dy(t)/dt=2y(t)+2t$ (vi) $dy(t)/dt=2y(t)+2t^2$.

4. Solve the following differential equations. Definitize the solutions using $y(0)=1$ when $t=0$.

(i) $dy(t)/dt=y(t)/t$ (ii) $dy(t)/dt=t^2/y(t)^2$ (iii) $dy(t)/dt=y(t)+y(t)^2$
(iv) $dy(t)/dt=y(t)+y(t)^2$ (v) $dy(t)/dt=ty(t)+y(t)^2$ (vi) $dy(t)/dt=y(t)+ty(t)^2$.

5. Find the steady-states (if any) for the following differential equations and determine whether $y(t)$ converges to or diverges from the steady-states using phase diagrams.

(i) $dy(t)/dt = 2y(t) - y(t)^2$ (ii) $dy(t)/dt = 5 - 2y(t)$ (iii) $dy(t)/dt = 5y(t) - 2$
(iv) $dy(t)/dt = 5 - 1/y(t)$ (v) $dy(t)/dt = [3y(t)/4] - y(t)^2$ (vi) $dy(t)/dt = y(t)^2 - y(t)$.

6. **Application exercise**. Consider the problem in example 1 in Section 10.2.7. Suppose that the values are $C(0) = 100, Y(0) = 550, c = 0.5, \text{ and } m = 0.8$. Derive the differential equation in $Y(t)$ and solve it. Sketch the graph of the solution and draw the phase diagram for the differential equation. Does $Y(t)$ converge to, or diverge from, the steady-state (if any)?

7. **Application exercise**: Suppose that the quantities demanded and supplied of a good at time t are given by $q_d(t) = 25/p$ and $q_s(t) = p(t)$, respectively; where $p(t)$ denotes per unit price in dollars. Suppose also that the rate of change of price at time t is given by $dp(t)/dt = [q_d(t) - q_s(t)]$ and that the market clears always. Draw the phase diagram for the differential equation and determine whether $p(t)$ converges to or diverges from the steady-state if it exists.

8. **Application exercise**. Suppose that the rate of change of a sum of money is given by the function $dF(t)/dt = rF(t)$, where $F(t)$ *and* r denote the sum of money at time t and rate of interest (compounded continuously), respectively. How long will it take for the initial sum $F(0) = 100$ to triple if the interest rate is $r = 8$ percent?

9. **Application exercise**. Assume that the point elasticity of demand for a good is given by $\Phi = -\alpha$, where α is a constant. Find the demand function for the good.

10. **Application exercise**. Assume that the marginal revenue at time t of selling $x(t)$ units of a good when the total revenue $y(t)$ is given by the function $dy(t)/dx(t) = x(t)/y(t)^2$. Determine the total revenue function $y(t)$ assuming that the constant of integration is equal to 1.

12. **Application exercise**. Assume that the quantities demanded and supplied of a good at time t are given by $q_d(t) = 80 - 4p(t) + 4[dp(t)/dt]$ and $q_s(t) = s_0 + 6p(t) + 6[dp(t)/dt]$, respectively; where $q_d(t), q_s(t) \text{ and } p(t)$ denote the quantity demanded, the quantity supplied, and the per unit price of the good, respectively. Determine the time path of price assuming that $p(0) = 10$ or $p(0) = 5$.

13. **Application exercise**. Consider the continuous time version of the Solow model of economic growth solved in example 7 in Section 10.2.7. Assume that the conditions of the problem there remain the same except that the rate of change of capital is now given by $dK(t)/dt = sY(t) - \delta K(t)$, where δ denotes the constant, positive **depreciation rate**. Derive the differential equation in $k(t)$. Use a phase diagram to draw inference about the intertemporal behavior of $k(t)$.

10.2.9 Mathematica applications

Differential equations can be solved in Mathematica using the built-in-command **DSolve**. The command **DSolve[*eqn*,*y*,*t*]** finds the solution to the differential equation ***eqn*** for the function ***y***, with independent variable *t*. As demonstrations, consider the differential equations we solved in Section 10.2.2: $dy(t)/dt = 0.5y(t) + 1$ and $dy(t)/dt = -0.5y(t) + 5$. These differential equations can be solved in Mathematiaca using the command **DSolve**, to obtain the same results as those we obtained in Section 10.2.2, as

```
DSolve[y'[t]==0.5y[t]+1,y[t],t]
DSolve[y'[t]==-0.5y[t]+5,y[t],t]
{{y[t]→-2. ⅇ0. t+ⅇ0.5 t C[1]}}
{{y[t]→10. ⅇ0. t+ⅇ-0.5 t C[1]}}
```

We can also use initial conditions with the command **DSolve**. Suppose that we use the initial conditions $y(0) = 1$ and $y(0) = 11$ with the differential equations $dy(t)/dt = 0.5y(t) + 1$ and $dy(t)/dt = -0.5y(t) + 5$ respectively. Then these can be solved as

```
DSolve[{y'[t]==0.5y[t]+1,y[0]==1},y[t],t]
DSolve[{y'[t]==-0.5y[t]+5,y[0]==11},y[t],t]
{{y[t]→-2.+3. ⅇ0.5 t}}
{{y[t]→ⅇ-0.5 t (1. +10. ⅇ0.5 t)}}
```

10.3 Higher-Order Linear Differential Equations

10.3.1 Introduction

In Section 10.2 we were concerned with the first-order derivative or differential only. In other words, we were concerned with the relationship between $dy(t)/dt$ and $y(t)$. As in the case of difference equations, it must be emphasized that differential equations are not restricted to first-order. They may, in fact, assume any order. One can cite a number of examples of higher order differential equations from the subjects of our interest. The continuous time version of the long-run Philips curve considered in example 3 in Section 9.3.7, the model of price determination in example 5 in Section 10.2.7, etc. are just a few to cite. This point to the fact that the students of economics, business and finance need to learn the solution and other topics related to the second-and-higher-order differential equations. Therefore, we turn our attention to such equations in the present section. The reader will see that most of our discussion in this section will parallel our discussion in Section 9.3.

10.3.2 Second-order, linear, autonomous differential equations

To begin with, consider the general second-order, linear, autonomous differential equation

$$d^2y(t)/dt^2 = a_1[dy(t)/dt] + a_2y(t) + b, \qquad (10.3.1)$$

which is identical with equation (10.2.3). Notice that equation (10.3.1) is second-order because the highest derivative in the equation is a second-order derivative; is linear because none of the $y(t)$ term is raised to a power other than 1 and is not multiplied or divided by any other $y(t)$ term; and is autonomous because the coefficients a_1 *and* a_2 and the term b are constants.

Our aim in this section is to find the solution to equation (10.3.1). The method of solution to the first-order, linear, autonomous differential equations discussed in Section 10.2 can lead us in finding this. As in the case of equation (10.2.11), the complete solution to equation (10.3.1) also comprises of two parts: the complementary function and the particular solution. Therefore, let us first find the particular solution before moving to the determination of the complementary function.

Recall that the particular solution in the case of the general first-order, linear, autonomous differential equation (10.2.1) was given in equation (10.2.10): $y(s) = p = -b/a$. This was derived on the assumption that $y(t) = p$ and $dy(t)/dt = d(p)/dt = 0$. We can follow the same procedure in the case of the present problem too. Suppose that $y(t) = p$, which implies that, since p is a constant, $dy(t)/dt = d(p)/dt = 0$ and $d^2y(t)/dt^2 = d^2(p)/dt^2 = 0$. Substituting these into equation (10.3.1) we obtain $0 = a_1 0 + a_2 p + b$ or $p = b/-a_2$, where $a_2 = 0$. Therefore, following the derivation of equation (10.2.10), the particular solution in the present case becomes

$$y(s) = p = b/(-a_2). \qquad (10.3.2)$$

If $a_2 = 0$ our trial solution will not work and, therefore, we will have to use trial solutions, as in the case of difference equations, such as $y(t) = pt$, $y(t) = pt^2$, etc.

Having determined the particular solution, we now attempt to determine the complementary function of the homogenous form ($b = 0$) of the differential equation. Recall that we obtained the solution to equation (10.2.1) as $y(c) = Ce^{at}$ and called it the complementary function. Let us write, for convenience, the last equation as $y(c) = Ce^{vt}$, where $v = a$. We may, therefore, assume that $y(c) = Ce^{vt}$ in the present case too, which implies that $dy(c)/dt = Cve^{vt}$ and $d^2y(c)/dt^2 = Cvve^{vt} = Cv^2e^{vt}$. Substituting these into the homogenous but alternative form of equation (10.3.1) $d^2y(t)/dt^2 - a_1dy(t)/dt - a_2y(t) = 0$, we obtain $Cv^2e^{vt} - a_1Cve^{vt} - a_2Ce^{vt} = 0$, or $Ce^{vt}[v^2 - a_1v - a_2] = 0$. Since Ce^{vt} cannot be zero, the last equation implies

$$v^2 - a_1v - a_2 = 0, \tag{10.3.3}$$

which is a quadratic equation, similar to equation (1.6.8). This quadratic equation is called the characteristic equation of the second-order, linear, homogenous differential equation $d^2y(t)/dt^2 - a_1dy(t)/dt - a_2y(t) = 0$. We can apply the quadratic formula in equation (1.6.9) to find the characteristic roots of equation (10.3.3). Since equation (10.3.3) is a quadratic equation, it possesses two roots and applying the quadratic formula we obtain the roots as

$$v_1, v_2 = \frac{-a_1 \pm \sqrt{a_1^2 - 4 \times 1 \times a_2}}{2 \times 1} = \frac{-a_1 \pm \sqrt{a_1^2 - 4a_2}}{2} = \frac{-a_1}{2} \pm \frac{\sqrt{a_1^2 - 4a_2}}{2}, \tag{10.3.4}$$

which is identical with equation (9.3.4). Therefore, as in the case of difference equations, the reader must have noticed that there may arise three different situations with regard to the roots depending upon the value of square-root expression in equation (10.3.4). We may obtain real and distinct roots; real and repeated roots; or complex roots. Each of these is explained below.

Real and distinct roots

Suppose that $a_1^2 > 4a_2$. Then we will obtain two real and distinct values for the roots v_1 *and* v_2. In this event, expressions v_1t and v_2t [in $y(c) = Ce^{v_1t}$ and $y(c) = Ce^{v_2t}$, respectively] will be linearly independent. Therefore, the complementary function becomes

$$y(c) = C_1e^{v_1t} + C_2e^{v_2t}, \tag{10.3.5}$$

where C_1 *and* C_2 are constants. Therefore, we can now write the complete solution to equation (10.3.1) by combining the complementary function [equation (10.3.5)] and the particular solution [equation (10.3.2)] as

$$y(t) = y(c) + y(s) = C_1e^{v_1t} + C_2e^{v_2t} + [b/(-a_2)] \tag{10.3.6}$$

where the values of the constants C_1 *and* C_2 cab determined through initial conditions.

Real and repeated roots

Suppose that $a_1^2 = 4a_2$. Then equation (10.3.4) implies that $v_1, v_2 = -a_1/2$; that is, the two roots are real and repeated (or equal). Since $v_1, v_2 = -a_1/2$, we can write $v = v_1 = v_2 = -a_1/2$. Then we may be tempted to write the complementary function, as we did in the case of equation (10.3.5) and as noted in Section 9.3.2, as $y(c) = C_1e^{v_1t} + C_2e^{v_2t} = (C_1 + C_2)e^{v_1t} = (C_1 + C_2)e^{vt} = C_3e^{vt}$, where $C_3 = C_1 + C_2$. But, although we need two constants, this complementary function contains only one constant C_3 and, therefore, it will not constitute a solution to the homogenous form of equation (10.3.1). One way out of this problem, as we did in the case of difference equations in Section 9.3.2, is to use an additional term of the form C_4te^{vt}, where C_4 is a constant. Adding this term to $y(c) = C_3e^{vt}$, we obtain the complementary function

$$y(c) = C_3e^{vt} + C_4te^{vt}. \tag{10.3.7}$$

Therefore, we can now write the complete solution to equation (10.3.1) by combining the complementary function [equation (10.3.7)] and the particular solution [equation (10.3.2)] as

$$y(t) = y(c) + y(s) = C_3 e^{vt} + C_4 t e^{vt} + [b/(-a_2)]. \tag{10.3.8}$$

As before, we can determine the values of the constants C_3 *and* C_4 through initial conditions.

Complex roots

As the last case, suppose that $a_1^2 < 4a_2$. In this event, the roots become complex numbers. Notice that we can write equation (10.3.4) as $v_1, v_2 = (-a_1/2) \pm [\sqrt{(4a_2 - a_1^2) \times -1}]/2 = (-a_1/2) \quad \pm [\sqrt{(4a_2 - a_1^2)}\sqrt{-1}]/2 =$ $(-a_1/2) \pm [\sqrt{(4a_2 - a_1^2)} \times i]/2$, where $i = \sqrt{-1}$. Notice also that the equation has a real part $(-a_1/2)$ and an imaginary part ($[\sqrt{(4a_2 - a_1^2)} \times i]/2$), making v_1 *and* v_2 complex numbers. Instead of denoting the opposite side of a triangle by v as we did in Section S1.12, we denote it here by q to distinguish it from the characteristic root notation v and, thereby, to avoid notational confusion. Let us now denote the real part of the complex number by $u = -a_1/2$ and the imaginary part by $qi = [\sqrt{(4a_2 - a_1^2)} \times i]/2$, where $q = [\sqrt{(4a_2 - a_1^2)}]/2$. Therefore, the two complex roots can be written as $v_1, v_2 = u \pm qi$. This implies that the complementary function of the solution to equation (10.3.1) becomes

$$y(c) = C_1 e^{v_1 t} + C_2 e^{v_2 t} = C_1 e^{(u+qi)t} + C_2 e^{(u-qi)t} \text{ or } y(c) = e^{ut}\left(C_1 e^{qit} + C_2 e^{-qit}\right), \tag{10.3.9}$$

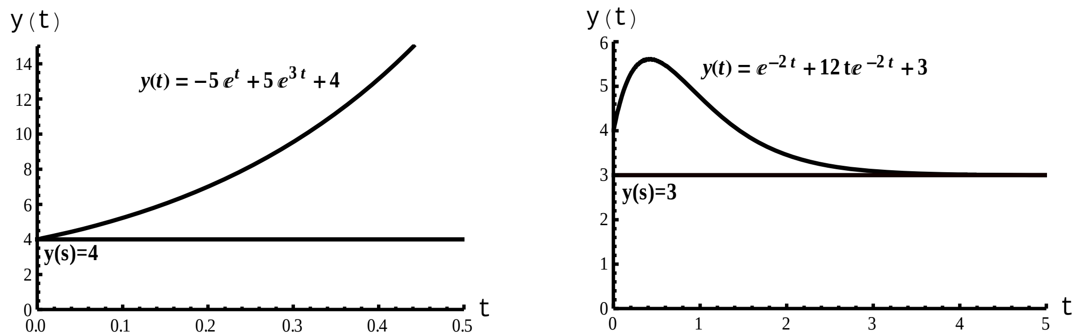
where C_1 *and* C_2 are constants. We can now apply Euler formulas in equations (S3.6.18) and (S3.6.19) to obtain $e^{qit} = \cos qt + i \sin qt$ and $e^{-qit} = \cos qt - i \sin qt$. This implies that we can write $C_1 e^{qit} + C_2 e^{-qit}$ in equation (10.3.9) as $C_1(\cos qt + i \sin qt) + C_2(\cos qt - i \sin qt)$. Therefore, equation (10.3.9) can be rewritten as $y(c) = e^{ut}[C_1(\cos qt + i \sin qt) + C_2(\cos qt - i \sin qt)]$; or $y(c) = e^{ut}[C_1 \cos qt + C_2 \cos qt + C_1 i \sin qt - C_2 i \sin qt]$; or $y(c) = e^{ut}[(C_1 + C_2)\cos qt + (C_1 - C_2) i \sin qt]$, which can be written as $y(c) = e^{ut}[C_3 \cos qt + C_4 \sin qt]$, where $C_3 = C_1 + C_2$ and $C_4 = (C_1 - C_2)i$. Therefore, the complementary function with complex roots becomes

$$y(c) = e^{ut}[C_3 \cos qt + C_4 \sin qt], \tag{10.3.10}$$

Having found the complementary function, we can now write the complete solution to equation (10.3.1) by combining the complementary function [equation (10.3.10)] and the particular solution [equation (10.3.2)] as

$$y(t) = y(c) + y(s) = e^{ut}(C_3 \cos qt + C_4 \sin qt) + [b/(-a_2)]. \tag{10.3.11}$$

As an example, consider the equation $d^2 y(t)/dt = 4dy(t)/dt - 3y(t) + 12$, which is identical with equation (10.3.1) for $a_1 = 4, a_2 = -3,$ *and* $b = 12$. Since $a_2 = -3 \neq 0$, we can use equation (10.3.2) to obtain the particular solution: $y(s) = p = b/-a_2 = 12/-(-3) = 4$. Let us now attempt to find the complementary function. Assume that we use the trial solution $y(c) = Ce^{vt}$, which implies that $dy(t)/dt = Cve^{vt}$ and $d^2 y(c)/dt = Cvve^{vt} = Cv^2 e^{vt}$. With these results we can write the homogenous form of equation as $Cv^2 e^{vt} - 4Cve^{vt} + 3Ce^{vt} = 0$ or $Ce^{vt}[v^2 - 4v + 3] = 0$. Then the characteristic equation of the differential equation can be written in the form of equation (10.3.3): $v^2 - 4v + 3 = 0$. Applying equation (10.3.4), we obtain the characteristic roots or eigenvalues $v_1, v_2 = 1, 3$, which are real and distinct. Therefore, we can apply equation (10.3.5) to find the complementary function: $y(c) = C_1 e^{v_1 t} + C_2 e^{v_2 t} = C_1 e^{1t} + C_2 e^{3t}$. We shall now combine the particular solution with the complementary function to obtain the complete solution in the form of equation (10.3.6): $y(t) = y(c) + y(s) = C_1 e^{v_1 t} + C_2 e^{v_2 t} + b/-a_2 = C_1 e^{t} + C_2 e^{3t} + 4$. This solution can be definitized with $y(t) = 4$ and $dy(t)/dt = 10$ when $t = 0$ to obtain the complete and definite solution $y(t) = -5e^{t} + 5e^{3t} + 4$, where $-5 = C_1$ *and* $5 = C_2$. The graph of this solution is illustrated in Panel (A) of Figure 10.3.1.



(A) (B)

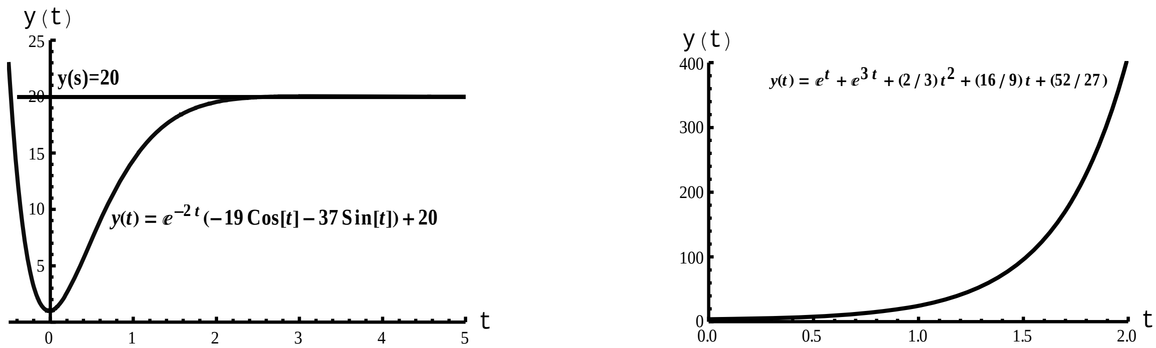
Figure 10.3.1

As another example, consider the equation $d^2y(t)/dt = -4dy(t)/dt - 4y(t) + 12$, which is identical with equation (10.3.1) for $a_1 = -4, a_2 = -4,$ *and* $b = 12$. Since $a_2 = -4 \neq 0$, we can use equation (10.3.2) to obtain the particular solution: $y(s) = p = b/-a_2 = 12/-(-4) = 3$. Let us now attempt to find the complementary function. Following the same procedure as before, the eigenvalues associated with the differential equation can be obtained as $v_1, v_2 = -2, -2$, which are real and repeated (equal). Therefore, we can apply equation (10.3.7) to find the complementary function: $y(c) = C_3e^{vt} + C_4te^{vt} = C_3e^{v_1t} + C_4te^{v_2t} = C_3e^{-2t} + C_4te^{-2t}$. Let us now combine the particular solution with the complementary function to obtain the complete solution in the form of equation (10.3.8): $y(t) = y(c) + y(s) = C_3e^{v_1t} + C_4te^{v_2t} + b/-a_2 = C_3e^{-2t} + C_4te^{-2t} + 3$. This solution can be definitized using $y(t) = 4$ and $dy(t)/dt = 10$ when $t = 0$, which give the complete and definite solution $y(t) = e^{-2t} + 12te^{-2t} + 3$, where $1 = C_3$ and $12 = C_4$. The graph of this solution is illustrated in Panel (B) of Figure 10.3.1.

As the last example, consider the equation $d^2y(t)/dt = -4dy(t)/dt - 5y(t) + 100$, which is identical with equation (10.3.1) for $a_1 = -4, a_2 = -5,$ *and* $b = 100$. Since $a_2 = -5 \neq 0$, the particular solution can be found using equation (10.3.2): $y(s) = p = b/-a_2 = 100/-(-5) = 20$. Following the same procedure as before, the eigenvalues associated with the differential equation can be obtained as $v_1, v_2 = -2 - i, -2 + i$ or $v_1, v_2 = u \pm qi$, where $u = -a_1/2 = -4/2 = -2$ and $q = \sqrt{(4a_2 - a_1^2)}/2 = \sqrt{(4 \times 5 - 16)}/2 = \sqrt{4}/2 = 1$. Notice that the roots are complex numbers. This suggests that we can apply equation (10.3.10) to find the complementary function: $y(c) = e^{ut}[C_3 \cos qt + C_4 \sin qt] = e^{-2t}[C_3 \cos t + C_4 \sin t]$. The complete solution can be obtained by combining the particular solution with the complementary function: $y(t) = y(c) + y(s) = e^{ut}(C_3 \cos qt + C_4 \sin qt) + b/-a_2 = e^{-2t}(C_3 \cos t + C_4 \sin t) + 20$. This solution can be definitized using $y(t) = 1$ and $dy(t)/dt = 1$ when $t = 0$. Thus the complete and definite solution becomes $y(t) = e^{-2t}(-19\cos t - 37\sin t) + 20$, where $-19 = C_3$ and $-37 = C_4$. The graph of this solution is illustrated in Panel (A) of Figure 10.3.2.

10.3.3 Convergence of second-order, linear, autonomous differential equations

We know from our exposition of first-order, linear, autonomous differential equations that their solutions consisted of two parts: the complementary function [$y(c)$] and the particular solution [$y(s)$]. Therefore, we denoted the complete solution to a general difference equation by the equation $y(t) = y(c) + y(s)$, where $y(s)$ denoted the steady-state value. In other words, we attempted to see whether $y(t) \to y(s)$ as $t \to \infty$. This suggests that we treated $y(s)$ as the reference point for convergence or divergence. Similarly, the solution to a general homogenous differential equation was denoted by $y(t) = y(c)$ because $y(s) = 0$ in its solution (as $b = 0$). These results are equally valid in the case of second-order, linear, autonomous differential equations too.



(A) (B)

Figure 10.3.2

One of our major inference, as stated in Section 10.2.3, was that convergence or divergence of the path of $y(t)$ depended on the sign of the coefficient, a , of the equation. If the sign of the coefficient a is negative, then the time path will converge; and if the sign of a is positive, the time path will diverge. These results are valid in the case of a second-order, linear, autonomous differential equation too. But, in the case of such equation, we have to take into account the signs of two roots (v_1 *and* v_2) and not the sign of just one (a). As we will see below, $y(t)$ will converge to or diverge from the steady-state value if the signs of the roots (irrespective of their being real and distinct, real and repeated, or complex) are negative and will diverge if they are positive. This steady-state value will be $y(s) = b/-a_2$ if $b \neq 0$ and $y(s) = 0$ or converge to the horizontal axis if $b = 0$.

We observed that there would be two characteristic roots (v_1 *and* v_2) in the case of a second-order differential equation and that these roots could be real and distinct roots, real and repeated roots, or complex roots. Let us first consider the case of real and distinct roots: $v_1 \neq v_2$. If the signs of both of these roots are positive or if $v_1 > 0$ *and* $v_2 > 0$, then the $y(c)$ part of equation (10.3.6) will increase as t increases implying that $y(t)$ will diverge from $y(s)$ if $b \neq 0$ or from the horizontal axis if $b = 0$. This is illustrated through the graph (for $b \neq 0$) in Panel (A) of Figure 10.3.1. If the signs of both roots are negative or if $v_1 < 0$ *and* $v_2 < 0$, the $y(c)$ part of equation (10.3.6) will decrease as t increases implying that $y(t)$ will converge to $y(s)$ if $b \neq 0$ or to the horizontal axis if $b = 0$. This is illustrated through the graph (for $b \neq 0$) in Panel (B) of Figure 10.3.1. If $v_1 < 0$ *and* $v_2 > 0$, then the term $C_1 e^{v_1 t}$ of $y(t)$ will decrease while the term $C_2 e^{v_2 t}$ of $y(t)$ will increase as t increases causing $y(c)$, and there by $y(t)$, to deviate or diverge from the steady-state $y(s)$ eventually.

We shall now consider the convergence of the solutions when the characteristic roots are real but repeated as in equation (10.3.8). If $v < 0$, both $C_3 e^{vt}$ *and* $C_4 te^{vt}$ terms of $y(c)$ will decrease as t increases (even if the second term, $C_4 te^{vt}$, is multiplied by t) and, therefore, $y(t)$ will converge to $y(s)$ if $b = 0$ or to the horizontal axis if $b = 0$. This is illustrated through the graph (for $b \neq 0$) in Panel (B) of Figure 10.3.1. It is needless to say that $y(t)$ will diverge if $v = v_1 = v_2 > 0$.

Lastly, we can consider the convergence of the solutions when the characteristic roots are complex numbers. For this we can use the general solution in equation (10.3.11). Notice that the terms inside the brackets of the solution, which is equal to $y(c)$, will give us fluctuating graphs. However, whether $y(t)$ will converge to or diverge from the steady-state value of $y(s)$ if $b \neq 0$ (or to or from horizontal axis if $b = 0$) or not is still determined by the exponent ut , where u is the real part of the complex root. If $u < 0$ ($u > 0$), then $y(t)$ will converge (diverge). This is illustrated through the graph (for $b \neq 0$) in Panel (A) of Figure 10.3.2.

10.3.4 Second-order, linear, non-autonomous differential equations

So far in the present section we were concerned with the solutions (and their convergence or divergence) to second-order, linear, autonomous differential equations. Let us now consider the solutions to second-order, linear, non-autonomous differential equations with variable terms.

As a general case, consider the equation (10.3.1): $d^2y(t)/dt = a_1dy(t)/dt + a_2y(t) + b$. If we assume that the term b in the last equation is a function of time t (and, therefore, the equation is non-autonomous), then it becomes as a differential equation with a variable term. Suppose that a form of such an equation is given by

$$d^2y(t)/dt = a_1dy(t)/dt + a_2y(t) + 2t^2, \tag{10.3.12}$$

where a_1 *and* a_2 are constants. Notice that we now have, instead of a constant b, a term that is a function of time (that is, $2t^2$) in the equation. Since the coefficients are still constants, the move from a constant term to a variable term will only affect the particular solution. This suggests that the complementary function still remains the same as in the case of equations with constant coefficients. This also suggests that we need only attempt to determine the new particular solution. But, how do we do this? Our expositions in the cases of difference and differential equations in Sections 9.3.4 and 10.2.2, respectively, can lead us here.

In the case of differential equations with constant terms, we used the trial solution $y(t) = p$, which implied $dy(t)/dt = 0$ and $d^2y(t)/dt = 0$, to find the particular solution. Since the term is a variable in the present case, let us apply the method of undetermined coefficients using the trial solution $y(t) = B_1t^2 + B_2t + B_3$, which implies that $dy(t)/dt = 2B_1t$ and $d^2y(t)/dt^2 = 2B_1$. Substituting the last three equations into equation (10.3.12), we obtain $2B_1 = a_12B_1t + a_2(B_1t^2 + B_2t + B_3) + 2t^2$; or $2B_1 - a_12B_1t - a_2(B_1t^2 + B_2t + B_3) + 2t^2$; or, by collecting the similar terms, $-a_2B_1t^2 - (a_2B_2 + 2a_1B_1)t + (2B_1 - a_2B_3) = 2t^2$. Now equate the coefficients of the similar terms on either sides of the last equation to obtain $-a_2B_1 = 2$, $-(a_2B_2 + 2a_1B_1) = 0,$ *and* $(2B_1 - a_2B_3) = 0$ and solve them simultaneously to yield the values of $B_1, B_2,$ *and* B_3. Suppose that these values are given by $B_1^*, B_2^*,$ *and* B_3^*. Therefore, the particular solution becomes

$$y(s) = B_1^*t^2 + B_2^*t + B_3^*, \tag{10.3.13}$$

which is a function of t. Therefore, as in the case of difference equations, it is referred to as a moving equilibrium. If $a_2 = 0$, then we will have to try forms such as $y(t) = t(B_1t^2 + B_2t + B_3) = B_1t^3 + B_2t^2 + B_3t$, $y(t) = t^2(B_1t^2 + B_2t + B_3) = B_1t^4 + B_2t^3 + B_3t^2$, etc. Let us now write the complete solution to equation (10.3.12) by combining the complementary function [equation (10.3.5) if the characteristic roots are real and distinct; equation (10.3.7) if the roots are real and repeated; or equation (10.3.10) if the roots are complex numbers] and the particular solution in equation (10.3.13) as

$$y(t) = y(c) + y(s) = y(c) + B_1^*t^2 + B_2^*t + B_3^*, \tag{10.3.14}$$

where the term $y(c)$ represents equations (10.3.5), or (10.3.7), or (10.3.10) depending upon the roots are real and distinct, or real and repeated, or complex, respectively.

As an example, consider the equation $d^2y(t)/dt = 4dy(t)/dt - 3y(t) + 2t^2$. The homogenous version of this differential equation is $d^2y(t)/dt = 4dy(t)/dt - 3y(t)$ or $d^2y(t)/dt - 4dy(t)/dt + 3y(t) = 0$, which has already been solved with solution $y(c) = C_1e^{1t} + C_2e^{3t}$ in the first example in Section 10.3.2. Since a variable term affects only the particular solution, let us attempt to find it through the method of undetermined coefficients exposed above. Following the same line of arguments as above, we use the trial solution $y(t) = B_1t^2 + B_2t + B_3$., which implies that $dy(t)/dt = 2B_1t + B_2$ and $d^2y(t)/dt^2 = 2B_1$. Substituting the last three equations into the given differential equation, we obtain $2B_1 = a_1(2B_1t + B_2) + a_2(B_1t^2 + B_2t + B_3) + 2t^2$, or $2B_1 = 4(2B_1t + B_2) - 3(B_1t^2 + B_2t + B_3) + 2t^2$, or $2B_1 - 4(2B_1t + B_2) + 3(B_1t^2 + B_2t + B_3) = 2t^2$, or by collecting the similar terms $3B_1t^2 + (3B_2 - 8B_1)t + (2B_1 - 4B_2 + 3B_3) = 2t^2$. We can now equate the coefficients of the similar terms on either sides of the last equation to obtain $3B_1 = 2$, $(3B_2 - 8B_1) = 0,$ *and* $(2B_1 - 4B_2 + 3B_3) = 0$, which can be solved simultaneously to yield $B_1^* = 2/3, B_2^* = 16/9,$ *and* $B_3^* = 52/27$. Using equation (10.3.13), therefore, the particular solution can be written as $y(s) = (2/3)t^2 + (16/9)t + (52/27)$. Then, using equation (10.3.14), the complete solution can be written as $y(t) = y(c) + y(s) = y(c) + B_1^* t^2 + B_2^* t + B_3^* = C_1 e^t + C_2 e^{3t} + (2/3)t^2 + (16/9)t + 52/27$, the graph of which assuming $C_1 = C_2 = 1$ is illustrated in Panel (B) of Figure 10.3.2.

As another example, consider the equation $d^2 y(t)/dt = -4dy(t)/dt - 4y(t) + 2t^2 + 2$. The homogenous version of this differential equation is $d^2 y(t)/dt = -4dy(t)/dt - 4y(t)$ or $d^2 y(t)/dt + 4dy(t)/dt + 4y(t) = 0$, which has already been solved with solution $y(c) = C_3 e^{-2t} + C_4 te^{-2t}$ in the second example in Section 10.3.2. Following the same procedure as earlier, let us use the trial solution $y(t) = B_1 t^2 + B_2 t + B_3$ implying that $dy(t)/dt = 2B_1 t + B_2$ and $d^2 y(t)/dt^2 = 2B_1$. Substituting the last three equations into the given differential equation, we obtain $2B_1 = a_1(2B_1 t + B_2) + a_2(B_1 t^2 + B_2 t + B_3) + 2t^2$, or $2B_1 = -4(2B_1 t + B_2) - 4(B_1 t^2 + B_2 t + B_3) + 2t^2 + 2$, or $2B_1 + (8B_1 t + 4B_2) + 4(B_1 t^2 + B_2 t + B_3) = 2t^2 + 2$, or by collecting the similar terms $4B_1 t^2 - (4B_2 + 8B_1)t + (2B_1 + 4B_2 + 4B_3) = 2t^2 + 2$. Equating the coefficients of the similar terms on either sides of the last equation yields $4B_1 = 2$, $(4B_2 + 8B_1) = 0,\ and\ (2B_1 + 4B_2 + 4B_3) = 2$, which can be solved simultaneously to yield $B_1^* = 0.5, B_2^* = -1,\ and\ B_3^* = 1.25$. Therefore, the particular solution can be written as $y(s) = 0.5t^2 - t + 1.25$. Then, using equation (10.3.14), the complete solution can be written as $y(t) = y(c) + y(s) = y(c) + B_1^* t^2 + B_2^* t + B_3^* = C_3 e^{-2t} + C_4 te^{-2t} + 0.5t^2 - t + 1.25$.

10.3.5 Higher-order, linear, autonomous differential equations: solutions

So far in the present chapter we were concerned with the solutions and their convergence of differential equations of order up to two. Let us now consider the solutions of equations of order higher than two. Consider first the general nth-order equation

$$\frac{d^n y(t)}{dt^n} = a_1 \frac{d^{n-1} y(t)}{dt^{n-1}} + a_2 \frac{d^{n-2} y(t)}{dt^{n-2}} + \dots + a_{n-1} \frac{dy(t)}{dt} + a_n y(t) + b \qquad (10.3.14)$$

As in the case of the complete solution to the first-order, linear, autonomous differential equation presented in equation (10.2.11), the solution to equation (10.3.14) also involves two parts: the complementary function and the particular solution. Let us first determine the complementary function of the solution to equation (10.3.14).

We first attempted the trial solutions earlier. We can follow the same trial solution in the case of the homogenous form of equation (10.3.14) too. Suppose that we try the solution $y(t) = Ce^{vt}$, which implies that $dy(t)/dt = Cve^{vt}$, $d^2 y(t)/dt^2 = Cv^2 e^{vt}$, $\dots, d^{n-1} y(t)/dt^{n-1} = Cv^{n-1} e^{vt}$, and $dy^n(t)/dt^n = Cv^n e^{vt}$. If we substitute these equations into the homogenous form of equation (10.3.14), we obtain $Cv^n e^{vt} = a_1 Cv^{n-1} e^{vt} + a_2 Cv^{n-2} e^{vt} + \dots + a_{n-1} Cve^{vt} + a_n Ce^{vt}$; or $Cv^n e^{vt} - a_1 Cv^{n-1} e^{vt} - a_2 Cv^{n-2} e^{vt} - \dots - a_{n-1} Cve^{vt} - a_n Ce^{vt} = 0$; or $Ce^{vt}[v^n - a_1 v^{n-1} - a_2 v^{n-2} - \dots - a_{n-1} v - a_n] = 0$. Since Ce^{vt} cannot be zero, then the last equation can be written as $[v^n - a_1 v^{n-1} - a_2 v^{n-2} - \dots - a_{n-1} v - a_n] = 0$, which is an nth-degree polynomial and is the characteristic equation of equation (10.3.14). If this characteristic equation is solved, we will obtain n characteristic roots or eigenvalues $v_1, v_2, \dots, v_n$. As we noticed in Section 10.3.2, these roots may be real and distinct, or real and repeated, or complex. If the roots are real and distinct, we can write the complementary function [following equation (10.3.5)] as

$$y(c) = C_1 e^{v_1 t} + C_2 e^{v_2 t} + \dots + C_n e^{v_n t} = \sum_{i=1}^{n} C_i e^{v_i t}, \qquad (10.3.15)$$

which is a generalization of equation (10.3.5). If the roots are real but repeated, the complementary function [following equation (10.3.7)] becomes

$$y(c) = C_1 e^{vt} + C_2 te^{vt} + \dots + C_n t^{n-1} e^{vt} = \sum_{i=1}^{n} C_i t^{i-1} e^{vt}. \qquad (10.3.16)$$

If any of the two characteristic roots, say v_{n-1} *and* v_n, happen to be complex, then we can apply equation (10.3.10) to obtain the complementary function corresponding to these complex roots as $y(c)_{(n-1),n} = e^{ut}(C_{n-1}\cos qt + C_n \sin qt)$, which can be combined with equation (10.3.15) to obtain

$$y_c = C_1 e^{v_1 t} + C_2 e^{v_1 t} + ... + C_{n-2} e^{v_{n-2} t} + e^{ut}[C_{n-1}\cos qt + C_n \sin qt]$$
$$= \sum_{i=1}^{n-2} C_i e^{v_i t} + e^{ut}[C_{n-1}\cos qt + C_n \sin qt] \cdot \qquad (10.3.17)$$

Let us now attempt to determine the particular solution using equation (10.3.14). In the case of both first-order and second-order, linear differential equations we used trial solutions such as $y(t) = p$, where p was a constant, to find the particular solutions in equations (10.2.10) and (10.3.2). We can use the same procedure to find the particular solution to equation (10.3.14). Suppose that we use the trial solution $y(t) = p$, which implies that $dy(t)/dt = 0$, $d^2 y(t)/dt^2 = 0$,…, and $d^n y(t)/dt^n = 0$. Substituting these values into equation (10.3.14) we obtain $0 = a_1 0 + a_2 0 + ... + a_{n-1} 0 + a_n p + b$; or $-a_n p = b$; or

$$y(s) = p = b/(-a_n), \qquad (10.3.18)$$

where $a_n \neq 0$. If $a_n = 0$, as stated in Sections 10.2.2 and 10.3.2, we will have to use trial solutions such as $y(t) = pt$, $y(t) = pt^2$, etc.

Since we obtained both the complementary function and the particular solution, we can now combine them to find the complete solution to equation (10.3.14) as

$$y(t) = y(c) + y(s) = \sum_{i=1}^{n} C_i v_i^t + [b/(-a_n)], \qquad (10.3.19)$$

if the roots are real and distinct; or as

$$y(t) = y(c) + y(s) = \sum_{i=1}^{n} C_i t^{i-1} e^{vt} + [b/(-a_n)], \qquad (10.3.20)$$

if the roots are real and repeated; or as

$$y(t) = y(c) + y(s) = \sum_{i=1}^{n-2} C_i e^{v_i t} + e^{ut}[C_{n-1}\cos qt + C_n \sin qt] + [b/(-a_n)], \qquad (10.3.21)$$

if two the roots are complex.

As an example, consider the third-order, linear, autonomous, non-homogenous differential equation $d^3 y(t)/dt^3 = -d^2 y(t)/dt^2 + dy(t)/dt + 0.5y(t) - 10$, which is a special case of equation (10.3.14) for $n = 3, a_1 = -1, a_2 = 1, a_3 = 0.5,$ *and* $b = -10$. The homogenous form of this equation is $d^3 y(t)/dt^3 = -d^2 y(t)/dt^2 + dy(t)/dt + 0.5y(t)$ or $d^3 y(t)/dt^3 + d^2 y(t)/dt^2 - dy(t)/dt - 0.5y(t) = 0$. The complementary function can be obtained using the trial solution $y(t) = Ce^{vt}$, which implies that $dy(t)/dt = Cve^{vt}$, $d^2 y(t)/dt^2 = Cv^2 e^{vt}$, and $d^3 y(t)/dt^3 = Cv^3 e^{vt}$. If we substitute the last four equations into the homogenous form of the differential equation, we obtain $Cv^3 e^{vt} + Cv^2 e^{vt} - Cve^{vt} - 0.5Ce^{vt} = 0$ from which the characteristic roots as can be obtained as $v_1 = -1.452, v_2 = -0.403,$ *and* $v_3 = 0.855$. These roots are real and distinct and, therefore, we can apply equation (10.3.15), with $n = 3$, to write the complementary function: $y(c) = C_1 e^{v_1 t} + C_2 e^{v_2 t} + C_3 e^{v_3 t} = C_1 e^{-1.452t} + C_2 e^{-0.403t} + C_3 e^{0.855t}$. The particular solution can be found by using equation (10.3.18). Since $a_3 = 0.5 \neq 0$, it becomes $y(s) = p = b/-a_3 = -10/-0.5 = 20$. Therefore, using equation (10.3.19), the complete solution can be written as $y(t) = y(c) + y(s) \quad = C_1 e^{-1.452\,t} + C_2 e^{-0.403\,t} + C_3 e^{0.855\,t} + 20$. Assuming, for simplicity, $C_1 = C_2 = C_3 = 1$, the solution can be written as $y(t) = e^{-1.452\,t} + e^{-0.403\,t} + e^{0.855\,t} + 20$.

10.3.6 Higher-order, linear, autonomous differential equations: convergence

We found from the solutions to differential equations so far that the time path given by the solutions to concerned variables would converge or diverge depending upon the signs of the characteristic root(s) of the equations. If the sign(s) of the root(s) is (are) negative, then the time path will converge to the steady-state if the constant term $b \neq 0$ or to zero or the horizontal axis if $b = 0$. If the sign(s) of the root(s) is (are) positive, then the time path will diverge from the steady-state if the constant term $b \neq 0$ or from zero or the horizontal axis if the constant term $b = 0$. In the case of characteristic equations with complex numbers as the roots, the sign mentioned above is the sign of the real part of the complex numbers. Therefore, if one knows the sign(s) of the root(s), one can infer whether the time path of the variable will converge or diverge.

Finding the characteristic roots is not a big problem if the characteristic polynomial or the characteristic equation, such as equation (10.3.3), is of degree 2 or 3; that is, in the case of second-order or third-order differential equations. But, the problem will become more and more cumbersome as the degree of the polynomial increases. The question, then, is how one can determine whether the time path of the variable implied by a higher-order differential equation converges or diverges. One does not need to find the actual roots and then determine whether the time path converges or not. One can obtain the same conclusion qualitatively using a theorem called **Routh-Hurwitz theorem**. Since the proof of this theorem is beyond the scope of this book, we simply state it as follows. Routh-Hurwitz theorem states that the real parts of the roots of an nth-degree polynomial equation such as

$$[v^n - a_1 v^{n-1} - a_2 v^{n-2} - ... - a_{n-1} v - a_n] = 0 \qquad (10.3.22)$$

will be negative (and then the time path of the variable of the differential equation will converge) if and only if the first n of the following determinants are all positive:

E₁$=|\,a_1\,|$, **E₂**$=\begin{vmatrix} a_1 & a_3 \\ a_0 & a_2 \end{vmatrix}$, **E₃**$=\begin{vmatrix} a_1 & a_3 & a_5 \\ a_0 & a_2 & a_4 \\ 0 & a_1 & a_3 \end{vmatrix}$, **E₄**$=\begin{vmatrix} a_1 & a_3 & a_5 & a_7 \\ a_0 & a_2 & a_4 & a_6 \\ 0 & a_1 & a_3 & a_5 \\ 0 & a_0 & a_2 & a_4 \end{vmatrix}$,…, and **E₅** $=\begin{vmatrix} a_1 & a_3 & a_5 & a_7 & ... & 0 \\ a_0 & a_2 & a_4 & a_6 & ... & 0 \\ 0 & a_1 & a_3 & a_5 & ... & 0 \\ 0 & 0 & a_2 & a_4 & ... & 0 \\ 0 & 0 & a_1 & a_3 & ... & 0 \\ ... & ... & ... & ... & ... & ... \\ 0 & 0 & 0 & 0 & ... & a_n \end{vmatrix}$.

Two points must be noted when one applies the above theorem. The first is that $a_0 = 1$; that is, the coefficient of the term with the highest order in the differential equation is 1. The second is that we need to set $a_j = 0$ for all $j > n$; that is, we need to set the non-existing coefficients in the differential equation to zero.

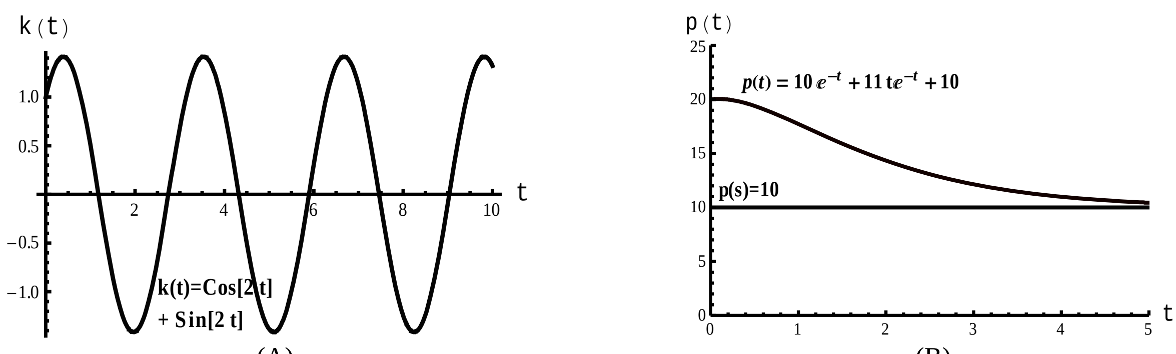
Let us now apply the convergence test using Routh-Hurwitz theorem. For this we use the second-order differential equation solved in the second example in Section 10.3.2. The homogenous form of this equation was $d^2 y(t)/dt = -4dy(t)/dt - 4y(t)$ or $d^2 y(t)/dt + 4dy(t)/dt + 4y(t) = 0$. We found there that the characteristic polynomial of this differential equation was $v^2 + 4v + 4 = 0$, which is identical with equation (10.3.22) if we treat $n = 2, a_0 = 1, a_1 = 4,$ *and* $a_2 = 4$. Since there are three coefficients ($a_0, a_1,$ *and* a_2) in the present differential equation, we need to use determinants up to $\mathbf{E_2}$ given above. Then, the determinants, with the above values for the coefficients and with $a_3 = 0$, become $\mathbf{E_1}$ = |4| = 4 > 0 and $\mathbf{E_2} = 4\times4 - 1\times0$ = 16 > 0. Since both determinants are positive, as per Routh-Hurwitz theorem, all the characteristic roots must be negative. Therefore, the path of the solution to the difference equation must converge. In fact, this was the result we found when we solved the equation and plotted the graph of the solution [$y(t) = e^{-2t} + 12te^{-2t} + 3$] in Panel (B) of Figure 10.3.1.

10.3.7 Application examples

Example 1: Suppose that the excess stock of capital, $K(t)$, at time t over its equilibrium level, $K(s)$, in an economy is given by $k(t) = K(t) - K(s)$. Also suppose that the rate of change of the excess stock of capital and that of the total investment, $I(t)$, in the economy are given by $dk(t)/dt = I(t)$ and $dI(t)/dt = -\alpha k(t)$, respectively, where $\alpha > 0$. Derive the implied differential equation and determine the time path of the excess stock of capital in the economy assuming $\alpha = 0$.

Solution: This is an example of a simplified version of **Samuelson's investment model**. Let us first derive the required differential equation by differentiating $dk(t)/dt = I(t)$ with respect to t to yield $d^2k(t)/dt^2 = dI(t)/dt$. We can now substitute $dI(t)/dt = -\alpha k(t)$ into the last equation to obtain $d^2k(t)/dt^2 = -\alpha k(t)$ or $d^2k(t)/dt^2 = -4k(t)$. The last equation is the required differential equation, which is identical with equation (10.3.1) for $k(t) = y(t), a_1 = 0, a_2 = \alpha,$ *and* $b = 0$. Since $b = 0$, the particular solution must be zero.

Let us now attempt to find the complementary function using the trial solution $k(t) = Ce^{vt}$, which implies that $dk(t)/dt = Cve^{vt}$ and $d^2k(t)/dt = Cv^2e^{vt}$. Substituting the last three equations into the differential equation we obtain $Cv^2e^{vt} = 0 \times Cve^{vt} - Ca_2e^{vt}$, or $Cv^2e^{vt} + Ca_2e^{vt} = 0$, or $Ce^{vt}[v^2 + a_2] = 0$. Following the same procedure as earlier, the two roots of the characteristic equation can be obtained as $v_1, v_2 = \pm\sqrt{\alpha} \times \sqrt{-1} = \pm\sqrt{a_2} \times \sqrt{-1} = \pm\sqrt{a_2} \times i$, where $i = \sqrt{-1}$. Notice that the roots are complex numbers. Following the notations introduced in Section 10.3.2, we can write these roots as $v_1, v_2 = u \pm qi$, where $u = -a_1/2$ and $q = \pm(\sqrt{a_1^2 - 4a_2})/2$. Since $a_1 = 0$, the last two results can be written as $u = -a_1/2$ and $q = \sqrt{a_2}$. Therefore, following equation (10.3.9), the complementary function can be written as $k(c) = C_1e^{v_1t} + C_2e^{v_2t} = C_1e^{(u+qi)t} + C_2e^{(u-qi)t} = C_1e^{qit} + C_2e^{-qit}$. Following the procedure adopted in the derivation of equation (10.3.10), we can write the complementary function as $k(c) = [C_3\cos qt + C_4\sin qt]$. Since $a_2 = 4$, and $q = \sqrt{a_2} = \sqrt{4} = 2$, the complementary function becomes $k(c) = [C_3\cos(\sqrt{a_2}t) + C_4\sin(\sqrt{a_2}t)]$, or $k(c) = [C_3\cos(2t) + C_4\sin(2t)]$. The last equation can be rewritten, assuming $C_3 = 1$ *and* $C_4 = 1$, as $k(c) = [\cos(2t) + \sin(2t)]$. We can now combine the particular solution and the complementary function, following equation (10.3.11), to find the complete and definite solution to the given differential equation as $k(t) = k(c) + k(s) = [C_3\cos(2t) + C_4\sin(2t)] + 0 = \cos(2t) + \sin(2t)$, the graph of which is illustrated in Panel (A) of Figure 10.3.3.



(A) (B)

Figure 10.3.3

Example 2: Consider the problem in example 2 in Section 10.2.7. Suppose that we include price expectations in the model. Suppose also that the demand and supply functions in the model, after expectations, take the forms $q_d(t) = 5 - 0.5p(t) - 2dp(t)/dt - d^2p(t)/dt^2$ and $q_s(t) = -5 + 0.5p(t)$, respectively. Set up the differential equation in price assuming the market clears and solve it. What is the intertemporal behavior of price if $p(0) = 20$ *and* $p'(0) = 1$?

Solution: First of all, let us derive the required differential equation. This can be done by equating the demand and supply functions: $5-0.5p(t)-2dp(t)/dt-d^2p(t)/dt^2=-5+0.5p(t)$ or $d^2p(t)/dt^2=-2dp(t)/dt-p(t)+10$, which is the required differential equation and it is identical with equation (10.3.1) for $p(t)=y(t), a_1=-2, a_2=-1, \text{ and } b=10$.

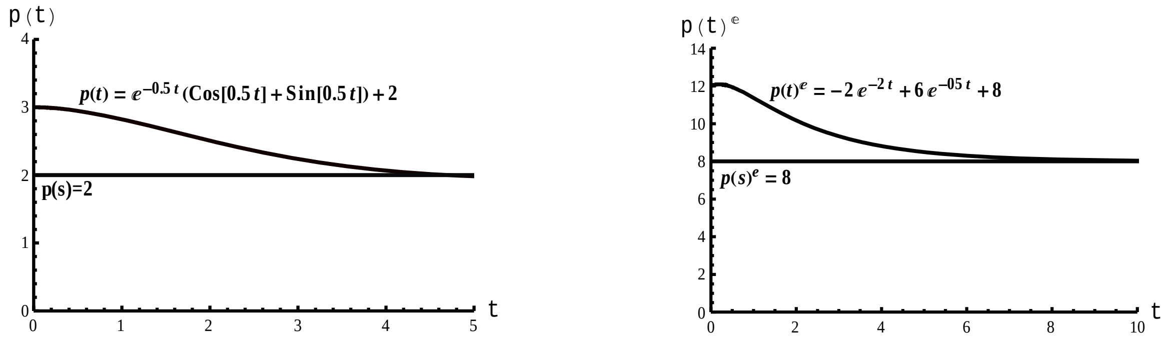
Particular solution of the differential equation can be found by applying equation (10.3.2): $p(s)=b/-a_2=10/-(-1)=10$. The complementary function can be found using the trial solution $p(t)=Ce^{vt}$, which implies that $dp(t)/dt=Cve^{vt}$ and $d^2p(t)/dt^2=Cv^2e^{vt}$. Substituting these three equations into the homogenous form of the differential equation gives $Cv^2e^{vt}=-2Cve^{vt}-Ce^{vt}$ or $Ce^{vt}[v^2+2v+1]=0$. The two roots of the associated characteristic equation can be obtained as $v_1, v_2=-1,-1$, which are real and equal. Then, applying equation (10.3.7), we obtain the complementary function as $p(c)=C_3e^{vt}+C_4te^{vt}=C_3e^{-t}+C_4te^{-t}$. Let us now combine the particular solution and the complementary function, as required by equation (10.3.8), to obtain the complete solution $p(t)=p(c)+p(s)=C_3e^{-t}+C_4te^{-t}+10$, which can be definitized, using the initial conditions $p(0)=20 \text{ and } p'(0)=1$, to obtain the complete and definite solution $p(t)=10e^{-t}+11te^{-t}+10$, where $10=C_3 \text{ and } 11=C_4$. The graph of solution is illustrated in Panel (B) of Figure 10.3.3.

Example 3: Suppose that a variant of the Walrasian price-adjustment model is such that the quantities demanded and supplied of a good at time t are given by $q_d(t)=15-5p(t)$ and $q_s(t)=-5+5p(t)$, respectively. Assume that the rate of change of the rate of change of price is given by $d^2p(t)/dt^2=0.1[(dq_d(t)/dt]-[dq_s(t)/dt)]-0.05[q_s-q_d]$. Set up the associated differential equation and solve it. Determine the intertemporal behavior of price assuming that the constants in the solution are equal to 1.

Solution: Let us first set up the required differential equation. For this we differentiate the demand and supply functions to obtain $dq_d(t)/dt=-5dp(t)/dt$ and $dq_s(t)/dt=5dp(t)/dt$, respectively. We can substitute these two equations, along with the demand and supply equations, into the above equation of the rate of change of the rate of change of price and simplify. The result will be $d^2p(t)/dt^2=-dp(t)/dt-0.5p(t)+1$, which is the required differential equation and it is identical with equation (10.3.1) for $p(t)=y(t), a_1=-1, a_2=-0.5, \text{ and } b=1$.

The particular solution to the differential equation can be found by applying equation (10.3.2): $p(s)=b/-a_2=1/-(-0.5)=2$. The complementary function can be found by using the trial solution $p(t)=Ce^{vt}$, which implies that $dp(t)/dt=Cve^{vt}$ and $d^2p(t)/dt^2=Cv^2e^{vt}$. Substituting these three equations into the homogenous form of the differential equation gives $Cv^2e^{vt}=-Cve^{vt}-0.5Ce^{vt}$, or $Ce^{vt}[v^2+v+0.5]=0$. The characteristic roots associated with the given differential equation can be obtained as $v_1, v_2=-0.5-0.5i, -0.5+0.5i$ or as $v_1, v_2=u\pm qi$, where $u=-a_1/2=-0.5$ and $q=[\sqrt{(4a_2-a_1^2)}]/2=0.5$. Notice that the roots are complex numbers. This implies that we can apply equation (10.3.10) to find the complementary function. Applying equation (10.3.10) yields the complementary function $p(c)=e^{ut}[C_3\cos qt+C_4\sin qt]$ $=e^{-0.5t}[C_3\cos(0.5t)+C_4\sin(0.5t)]$. Let us now combine the particular solution with the complementary function, as required by equation (10.3.11), to obtain the complete solution $p(t)=p(c)+p(s)=e^{ut}(C_3\cos qt+C_4\sin qt)+b/-a_2$ $=e^{-0.5t}[C_3\cos(0.5t)+C_4\sin(0.5t)]+2$. Definitizing the solution with $C_3=C_4=1$ gives the complete and definite solution $p(t)=e^{-0.5t}[\cos(0.5t)+\sin(0.5t)]+2$, the graph of which is illustrated in Panel (A) of Figure 10.3.4.

Example 4: Consider the **discrete time expectations-augmented Phillips curve** model solved in example 3 in Section 9.3.7. The **continuous time expectations-augmented Phillips curve** may be expressed as follows. In the continuous time version, the expectations-augmented Phillips curve; adaptive expectations; and monetary policy are represented by $p(t)=\alpha-\beta u(t)+p^e(t)$, $dp^e(t)/dt=h[p(t)-p^e(t)]$, and $du(t)/dt=-j[m-p(t)]$, respectively. In the last three equations $p(t), u(t), p^e(t), \text{ and } m$ denote the rate of inflation, the unemployment rate,



(A) (B)

Figure 10.3.4

the expected rate of inflation, and the rate of growth of the nominal money, respectively. It is assumed that $\alpha > 0, \beta > 0, j > 0,$ *and* $0 < h \le 1$, and $\alpha, \beta, j, h,$ *and* m are all constants. Determine the intertemporal behavior of the expected rate of inflation if $\alpha = 10, \beta = 0.5, h = 0.4, j = 10,$ *and* $m = 4$ and the initial conditions $p^e(0) = 12$ *and* $p^{e}{'}(0) = 1$.

Solution: Substituting $\alpha = 10, \beta = 0.5, h = 0.4, j = 10,$ *and* $m = 4$ into the above three equations they become $p(t) = 10 - 0.5u(t) + p^e(t)$, or $dp^e(t)/dt = 0.4[p(t) - p^e(t)] = 0.4p(t) - 0.4p^e(t)$, and $du(t)/dt = -10(4 - p(t)) = -40 + 10p(t)$. Substituting $p(t) = 10 - 0.5u(t) + p^e(t)$ into $dp^e(t)/dt = 0.4p(t) - 0.4p^e(t)$, the latter equation can be written as $dp^e(t)/dt = 0.4p(t) - 0.4p^e(t) = 0.4[10 - 0.5u(t) + p^e(t)] - 0.4p^e(t)$ and further simplified to obtain $dp^e(t)/dt = 4 - 0.2u(t)$. Differentiating the last result with respect to time yields $d^2p^e(t)/dt^2 = -0.2du(t)/dt$. Since $du(t)/dt = -40 + 10p(t)$, the last result becomes $d^2p^e(t)/dt^2 = -0.2du(t)/dt = -0.2[-40 + 10p(t)] = 8 - 2p(t)$. Notice that we can solve for $p(t)$ from $dp^e(t)/dt = 0.4p(t) - 0.4p^e(t)$ and obtain $p(t) = 2.5dp^e(t)/dt + p^e(t)$. Substituting this result into $d^2p^e(t)/dt^2 = 8 - 2p(t)$ and simplifying yields the required differential equation $d^2p^e(t)/dt^2 = -2.5dp^e(t)/dt - p^e(t) + 8$ and it is identical with equation (10.3.1) for $p^e(t) = y(t), a_1 = -2.5, a_2 = -1,$ *and* $b = 8$.

The particular solution can be found by applying equation (10.3.2): $p^e(s) = b/-a_2 = 8/-(-1) = 8$. The complementary function can be obtained using the trial solution $p^e(t) = Ce^{vt}$, which implies that $dp^e(t)/dt = Cve^{vt}$ and $d^2p^e(t)/dt^2 = Cv^2e^{vt}$. Substituting the last three equations into the homogenous form of the differential equation gives $Cv^2e^{vt} = -2.5Cve^{vt} - Ce^{vt}$, or $Ce^{vt}[v^2 + 2.5v + 1] = 0$. The last equation gives the characteristic roots $v_1, v_2 = -2, -0.5$, which are real and distinct. Then, applying equation (10.3.5), we obtain the complementary function: $p^e(c) = C_1e^{v_1t} + C_2e^{v_2t} = C_1e^{-2t} + C_2e^{-0.5t}$. We shall now combine the particular solution and the complementary function, as required by equation (10.3.6), to obtain the complete solution: $p^e(t) = p^e(c) + p^e(s) = C_1e^{-2t} + C_2e^{-0.5t} + 8$, which can be definitized using the initial conditions $p^e(0) = 12$ *and* $p^{e}{'}(0) = 1$. Therefore, the complete and definite solution becomes $p^e(t) = -2e^{-2t} + 6e^{-0.5t} + 8$, where $-2 = C_1$ *and* $6 = C_2$. The graph of this solution is illustrated in Panel (B) of Figure 10.3.4.

10.3.8 Exercises

1. Solve the following differential equations. Assume, for convenience, that all the constants in the solutions are equal to one.

(i) $d^2y(t)/dt^2 = -2dy(t)/dt$ (ii) $d^2y(t)/dt^2 = 2dy(t)/dt$ (iii) $d^2y(t)/dt^2 = 2dy(t)/dt + 3y(t)$
(iv) $d^2y(t)/dt^2 = -2dy(t)/dt - 2y(t) + 10$ (v) $d^2y(t)/dt^2 = -2dy(t)/dt - 4y(t) + 10$.

2. Solve the following differential equations. Assume, for convenience, that all the constants in the solutions are equal to one.

(i) $d^2 y(t)/dt^2 = dy(t)/dt + y(t) + t^2$ (ii) $d^2 y(t)/dt^2 = dy(t)/dt - y(t) + t^2$
(iii) $d^2 y(t)/dt^2 = -dy(t)/dt - y(t) + t^2 + t$ (iv) $d^2 y(t)/dt^2 = -2dy(t)/dt - y(t) + t^2 + t$.

3. Check whether $y(t)$ implied by the following differential equations converge or diverge. Assume, for convenience, that all the constants in the solutions are equal to one.
(i) $d^3 y(t)/dt^3 = d^2 y(t)/dt^2 + dy(t)/dt - y(t) + 10$ (ii) $d^3 y(t)/dt^3 = -d^2 y(t)/dt^2 + 10$
(iii) $d^3 y(t)/dt^3 = -dy(t)/dt + 10$ (iv) $d^3 y(t)/dt^3 = -y(t) + 10$.

4. **Application exercise**. Consider Samuelson's investment model solved in example 1 in Section 10.3.7. Assume that all the conditions in the model remain valid except that the rate of change of investment is given by $dI(t)/dt = -\alpha k(t) - 20$. Solve the model and determine the time path of the excess stock of capital assuming that the constants of the solution are equal to one.

5. **Application exercise**. Suppose that the differential equation in example 2 in Section 10.3.7 is $d^2 p(t)/dt^2 = -dp(t)/dt - 0.5p(t) + 2.5$. Determine the time path of price assuming that the constants of the solution equal one.

6. **Application exercise**. Assume that the differential equation in example 3 in Section 10.3.7 is $d^2 p(t)/dt^2 = -2dp(t)/dt - 0.5p(t) + 3$. Determine the time path of price assuming that the constants of the solution are equal to one.

7. **Application exercise**. Suppose that the differential equation in example 4 in Section 10.3.7 is $d^2 p^e(t)/dt^2 = -dp^e(t)/dt - 2p^e(t) + 8$. Determine the time paths of the rates of inflation and unemployment assuming that the constants of the solution are equal to one.

10.3.9 Mathematica applications

Higher-order differential equations can be solved in Mathematica using the command **DSolve**, used to solve first-order differential equations and introduced in Section 10.2. As a demonstration, consider the differential equations we solved in Section 10.3.2: $d^2 y(t)/dt = 4dy(t)/dt - 3y(t) + 12$; $d^2 y(t)/dt = -4dy(t)/dt - 4y(t) + 12$; and $d^2 y(t)/dt = -4dy(t)/dt - 5y(t) + 100$. Using the command **DSolve**, we can solve these differential equations without and with initial conditions to obtain the same results as those we obtained in Section 10.3.2 as

```
DSolve[y''[t]==4×y'[t]-3×y[t]+12,y[t],t]
DSolve[y''[t]==-4×y'[t]-4×y[t]+12,y[t],t]
DSolve[y''[t]==-4×y'[t]-5×y[t]+100,y[t],t]
{{y[t] → 4 + e^t C[1] + e^(3 t) C[2]}}
{{y[t] → 3 + e^(-2 t) C[1] + e^(-2 t) t C[2]}}
{{y[t] → 20 + e^(-2 t) C[2] Cos[t] + e^(-2 t) C[1] Sin[t]}}

DSolve[{y''[t]==4×y'[t]-3×y[t]+12,y[0]==4,y'[0]==10},y[t],t]
DSolve[{y''[t]==-4×y'[t]-4×y[t]+12,y[0]==4,y'[0]==10},y[t],t]
DSolve[{y''[t]==-4×y'[t]-5×y[t]+100,y[0]==1,y'[0]==1},y[t],t]
{{y[t] → 4 - 5 e^t + 5 e^(3 t)}}
{{y[t] → e^(-2 t) (1 + 3 e^(2 t) + 12 t)}}
{{y[t] → e^(-2 t) (20 e^(2 t) - 19 Cos[t] - 37 Sin[t])}}
```

10.4 Simultaneous Systems of Differential Equations

10.4.1 Introduction

So far in the present chapter we were concerned with the solutions to differential equations and the intertemporal behavior of these solutions. However, in all these analyses we were concerned with only one differential equation at a time. Our aim from now onwards in the present chapter is to solve, and determine the intertemporal behavior of these solutions, differential equations by taking more than one equation at a time; that is by taking **simultaneous systems of differential equations** (hereafter, **SSDLEs**).

As in the case of difference equations, one may wonder why we need to consider SSDLEs. Notice that in most of the differential equations we considered so far we assumed that the higher time derivative of a variable depended on either on the lower time derivative or the level, or both, of the same variable. Consider, for example, the general second-order, linear, differential equation $d^2y(t)/dt^2 = a_1 dy(t)/dt + a_2 y(t) + b$, which can be written in functional form as $d^2y(t)/dt^2 = y''(t) = f[dy(t)/dt, y(t)] = f[y'(t), y(t)]$. This equation shows that the second derivative of the variable y is a function of the first time derivative and the level of the same variable and no other variable is involved in the function.

But, it is easy to see that the above assumption, on the basis of which we framed the function $d^2y(t)/dt^2 = y''(t) = f[dy(t)/dt, y(t)] = f[y'(t), y(t)]$, is a restrictive one. The reason is that the higher time derivatives of many variables depend not only on the lower time derivatives or the levels of the same variables but also on the time derivatives or the levels of some other variable(s). Then we may get functions such as $y''(t) = g[y'(t), y(t), x(t)]$, $y''(t) = h[y'(t), y(t), x(t), x'(t)]$, $x''(t) = j[x'(t), x(t), y(t)]$, $x'' = k[x'(t), x(t), y'(t), y(t)]$, etc. One can cite many examples of this type of interdependence among variables over time in the fields of our interest. As an example, consider Samuelson's investment model solved in example 1 in Section 10.2.7. In this model we observed that the rate of change of excess capital stock depended on the level of investment and the rate of change of investment depended on the level of the excess stock of capital. As another example, consider the inflation and unemployment model solved in example 4 in Section 10.3.7. In this model we found that the rate of change of expected inflation rate depended on both unemployment rate and expected inflation rate; and the rate of change of unemployment rate, in turn, depended on both expected inflation rate and unemployment rate. These examples suggest that the differential equations involved are simultaneous in nature; or, in other words, we have SSDLEs.

The question, therefore, is: how can one solve such SSDLES and find out the time paths of the involved variables? As in the case of SSDEs, the techniques of solving SSDLEs are quite similar to those we used in solving single differential equations. There exist different methods to solve SSDLEs. We will expose only two methods here: the **substitution method** and the **matrix method**. These methods are quite similar to the substitution and matrix methods we employed to solve the SSDEs in the last chapter.

10.4.2 Solution of SSDLEs: the substitution method

Assume that we have the following SSDLEs:

$$dy(t)/dt = a_{11}y(t) + a_{12}x(t) + b_y, \quad \text{and} \quad dx(t)/dt = a_{21}y(t) + a_{22}x(t) + b_x \qquad (10.4.1)$$

In Sections 10.2.2 and 10.3.2 we solved the first-order and the second-and-higher-order differential equations, respectively, by finding and then combining the particular solutions of non-homogenous versions of the equations and the complementary functions of the homogenous versions of the equations. But, to find both particular solution and complementary function we used trial solutions. The solutions to SSDLEs also involve two parts: the particular solution and the complementary function.

Let us first determine the particular solution. As used in Sections 10.2.2 and 10.3.2, we can use trial solutions $y(t) = p_y(s)$ and $x_t = p_x(s)$, where $p_y(s)$ *and* $p_x(s)$ are constants. The last two equations imply that $dy(t)/dt = 0$ and $dx(t)/dt = 0$. Substitution of these equations into the system (10.4.1) yields $a_{11}p_y(s) + a_{12}p_x(s) + b_y = 0$ and $a_{21}p_y(s) + a_{22}p_x(s) + b_x = 0$, which can be solved simultaneously to obtain the particular solutions

$$y(s) = p_y(s) = \frac{a_{12}b_x - a_{22}b_y}{(a_{11}a_{22} - a_{12}a_{21})}, \text{ and} \qquad (10.4.2)$$

$$x(s) = p_x(s) = \frac{a_{21}b_y - a_{11}b_x}{(a_{11}a_{22} - a_{12}a_{21})}. \quad (10.4.3)$$

In case the trial solutions $y(t) = p_y(s)$ and $x(t) = p_x(s)$ fail to work or when the denominators in equations (10.4.2) and (10.4.3) happen to be zeros, then we have to use trial solutions such as $y(t) = p_y(s)t$ and $x(t) = p_x(s)t$; $y(t) = p_y(s)t^2$ and $x(t) = p_x(s)t^2$; etc.

Having found out the particular solutions, let us now attempt to find out the complementary functions. For this we use the homogenous version of the system (10.4.1):

$$dy(t)/dt = a_{11}y(t) + a_{12}x(t), \quad \text{and} \quad dx(t)/dt = a_{21}y(t) + a_{22}x(t). \quad (10.4.4)$$

Let us now differentiate the first equation in the system (10.4.4) with respect to time to obtain $d^2y(t)/dt^2 = a_{11}dy(t)/dt + a_{12}dx(t)/dt$. Substitution of the second equation in the system into the last result yields $d^2y(t)/dt^2 = a_{11}dy(t)/dt + a_{12}dx(t)/dt \quad = a_{11}dy(t)/dt + a_{12}a_{21}y(t) + a_{12}a_{22}x(t)$. Notice that we can solve for $x(t)$ from the first equation in the system as

$$x(t) = \{[dy(t)/dt] - a_{11}y(t)\}/a_{12}, \quad (10.4.5)$$

where it is assumed that $a_{12} \neq 0$. Substituting this value of $x(t)$ into $d^2y(t)/dt^2 = a_{11}dy(t)/dt + a_{12}a_{21}y(t) + a_{12}a_{22}x(t)$ we obtain, after simplification,

$$d^2y(t)/dt^2 = (a_{11} + a_{22})[dy(t)/dt] + (a_{12}a_{21} - a_{11}a_{22})y(t), \quad (10.4.6)$$

which is identical with equation (10.3.1) for $a_1 = (a_{11} + a_{22}), a_2 = (a_{12}a_{21} - a_{11}a_{22}), \text{ and } b = 0$. The reader must have noticed that when we moved from the system (10.4.4) to equation (10.4.6) we transformed a system of two, first-order, linear, autonomous, homogenous differential equations in two variables into a second-order, linear, autonomous, homogenous differential equation in one variable. This implies that we can, if wanted to, go back from equation (10.4.6) to equation (10.4.4). What all these mean is that we can always convert through substitution a simultaneous system of n first-order differential equations in n variables to an nth-order, linear, autonomous differential equation in one variable, and vice versa.

Let us now use the same method as those we used in Sections 10.2 and 10.3 to find the complementary function. Suppose that we use the trial solution $y(t) = Ce^{vt}$, which implies that $dy(t)/dt = Cve^{vt}$, and $d^2y(t)/dt^2 = Cv^2e^{vt}$. We can now substitute these equations into equation (10.4.6) to obtain $Cv^2e^{vt} = (a_{11} + a_{22})Cve^{vt} - (a_{11}a_{22} - a_{12}a_{21})Ce^{vt}$; or $Cv^2e^{vt} - (a_{11} + a_{22})Cve^{vt} + (a_{11}a_{22} - a_{12}a_{21})Ce^{vt} = 0$; or $Ce^{vt}[v^2 - (a_{11} + a_{22})v + (a_{11}a_{22} - a_{12}a_{21})] = 0$. We know that $Ce^{vt} \neq 0$ and, therefore, we have the result $v^2 - (a_{11} + a_{22})v + (a_{11}a_{22} - a_{12}a_{21}) = 0$, which can be solved to yield the characteristic roots $v_1 \text{ and } v_2$. If they are real and distinct, the complementary function becomes

$$y(c) = C_1e^{v_1t} + C_2e^{v_2t}. \quad (10.4.7)$$

We now need to find the complementary function for $x(t)$. This can be found by substituting equation (10.4.7) into equation (10.4.5). Once we do this and simplify, we obtain the complementary function for $x(t)$:

$$x(c) = \left[\frac{v_1 - a_{11}}{a_{12}}\right]C_1e^{v_1t} + \left[\frac{v_2 - a_{11}}{a_{12}}\right]C_2e^{v_2t}. \quad (10.4.8)$$

If the roots are real and repeated $(v = v_1 = v_2)$, then the complementary function for $y(t)$ becomes

$y(t) = C_1 e^{vt} + C_2 te^{vt}$, (10.4.9)

and the complementary function for $x(t)$ becomes

$$x(c) = \left[\frac{(v - a_{11})(C_1 + C_2 t)}{a_{12}}\right] e^{vt} + \left[\frac{C_2}{a_{12}}\right] e^{vt}. \quad (10.4.10)$$

If the roots are complex, then the complementary function for $y(t)$ becomes

$$y(c) = e^{ut}[C_1 \cos(qt) + C_2 \sin(qt)], \quad (10.4.11)$$

and the complementary function for $x(t)$ becomes

$$x(c) = e^{ut}\left[\left(\frac{C_1(u - a_{11}) + C_2 q}{a_{12}}\right)\cos(qt)\right] + e^{ut}\left[\left(\frac{C_2(u - a_{11}) - C_1 q}{a_{12}}\right)\sin(qt)\right]. \quad (10.4.12)$$

Let us now combine the particular solutions [equations (10.4.2) and (10.4.3)] with the complementary functions to obtain the complete solutions. If the roots are real and distinct, the complete solutions are

$y(t) = y(c) + y(s) = C_1 e^{v_1 t} + C_2 e^{v_2 t} + y(s)$, and (10.4.13)

$$x(t) = x(c) + x(s) = \left[\frac{v_1 - a_{11}}{a_{12}}\right] C_1 e^{v_1 t} + \left[\frac{v_2 - a_{11}}{a_{12}}\right] C_2 e^{v_2 t} + x(s).$$

If the roots are real and repeated, the complete solutions are

$y(t) = y(c) + y(s) = C_1 e^{vt} + C_2 te^{vt} + y(s)$, and (10.4.14)

$$x(t) = x(c) + x(s) = \left[\frac{(v - a_{11})(C_1 + C_2 t)}{a_{12}}\right] e^{vt} + \left[\frac{C_2}{a_{12}}\right] e^{vt} + x(s).$$

If the roots are complex, the complete solutions are

$y_t = y_c + y_s = e^{ut}[C_1 \cos(qt) + C_2 \sin(qt)] + y(s)$, and (10.4.15)

$$x(t) = x(c) + x(s) = e^{ut}\left[\left(\frac{C_1(u - a_{11}) + C_2 q}{a_{12}}\right)\cos(qt)\right] + e^{ut}\left[\left(\frac{C_2(u - a_{11}) - C_1 q}{a_{12}}\right)\sin(qt)\right] + x(s)$$

As an example, consider the SSDLEs:

$dy(t)/dt = -2y(t) - 2x(t) + 80$, and $dx(t)/dt = 0 \times y(t) - x(t) + 20$. (10.4.16)

Notice that the system (10.4.16) is identical with the system (10.4.1) for $a_{11} = -2, a_{12} = -2, a_{21} = 0, a_{22} = -1, b_y = 80,$ *and* $b_x = 20$. Let us first find the particular solutions. Using equations (10.4.2) and (10.4.3) for $a_{11} = -2, a_{12} = -2, a_{21} = 0, a_{22} = -1,\ b_y = 80,$ *and* $b_x = 20$ we obtain the particular solutions: $y(s) = p_y(s) = (a_{12}b_x - a_{22}b_y)/(a_{11}a_{22} - a_{12}a_{21})$ $=[-2\times 20-(-1)80]/[-2\times -1-(-2\times 0)] = 40/2 = 20,$ and $x(s) = p_x(s) =$ $(a_{21}b_y - a_{11}b_x)/(a_{11}a_{22} - a_{12}a_{21})$ $=[0\times 80-(-2)20]/[-2\times -1-(-2\times 0)] = 40/2 = 20$.

Let us now differentiate the first equation in the system (10.4.16) with respect to time to obtain $d^2y(t)/dt^2 = -2dy(t)/dt - 2dx(t)/dt$. Substitution of the homogenous form of the second equation in the system (10.4.16) into the last result yields $d^2y(t)/dt^2 = -2dy(t)/dt - 2dx(t)/dt$

$= -2dy(t)/dt - 2[0 \times y(t) - x(t)] = -2dy(t)/dt + 2x(t)$. Notice that we can obtain $x(t)$ from the homogenous form of the first equation in the system (102.4.16): $x(t) = [dy(t)/dt + 2y(t)]/-2$. Substituting this value of $x(t)$ into $d^2y(t)/dt^2 = -2dy(t)/dt + 2x(t)$ we obtain, after simplification, $d^2y(t)/dt^2 = -2dy(t)/dt + 2\{[(dy(t)/dt) + 2y(t)]/-2\}$; or $d^2y(t)/dt^2 = -2dy(t)/dt - dy(t)/dt - 2y(t) = -3dy(t)/dt - 2y(t)$, which is identical with equation (10.3.1) with $a_1 = -3, a_2 = -2,$ *and* $b = 0$.

To find the complementary function let us use the trial solution $y(t) = Ce^{vt}$, which implies that $dy(t)/dt = Cve^{vt}$ and $d^2y(t)/dt^2 = Cv^2e^{vt}$. We now substitute these equations into the derived differential equation to obtain $Cv^2e^{vt} = -3Cve^{vt} - 2Ce^{vt}$; or $Cv^2e^{vt} + 3Cve^{vt} + 2Ce^{vt} = 0$; or $Ce^{vt}[v^2 + 3v + 2] = 0$. The associated characteristic roots are $v_1 = -2$ *and* $v_2 = -1$, which are real and distinct and, therefore, we can apply equation (10.4.7) to find the complementary function for $y(t)$. Therefore, the complementary function for $y(t)$ becomes $y(c) = C_1e^{v_1t} + C_2e^{v_2t} = C_1e^{-2t} + C_2e^{-t}$. The complementary function for $x(t)$ can be found by applying equation (10.4.8): $x(c) = [(v_1 - a_{11})/a_{12}]C_1e^{v_1t} + [(v_2 - a_{11})/a_{12}]C_2e^{v_2t} = \{[-2-(-2)]/-2\}C_1e^{-2t} + \{[-1-(-2)]/-2\}C_2e^{-t} = -0.5C_2e^{-t}$. We can combine these complementary functions with the particular solutions obtained above to obtain the complete solutions for $y(t)$ and $x(t)$ as $y(t) = y(c) + y(s) = C_1e^{-2t} + C_2e^{-t} + 20$ and $x(t) = x(c) + x(s) = -0.5C_2e^{-t} + 20$, respectively. Using the initial conditions $y(0) = 20$ *and* $x(0) = 10$ when $t = 0$, the complete solutions can be definitized to yield $C_1 = -20$ *and* $C_2 = 20$. Thus the complete and definite solutions are $y(t) = -20e^{-2t} + 20e^{-t} + 20$ and $x(t) = -10e^{-t} + 20$. The graphs of these solutions are illustrated in Figure 10.4.1.

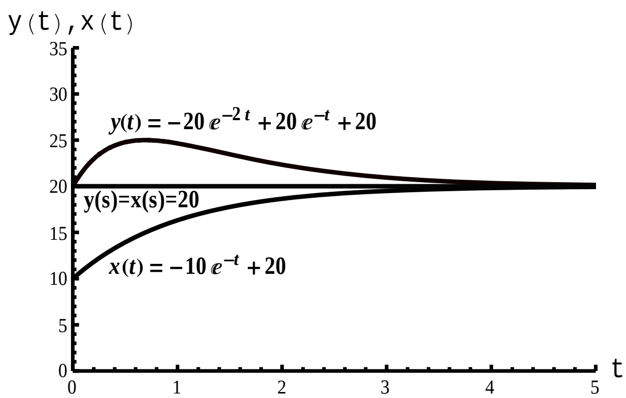


Figure 10.4.1

10.4.3 Solution of SSDLEs: the matrix method

In the last section we found how we could solve first-order, linear, autonomous SSDLEs in two variables by first transforming them, using the substitution method, into second-order, linear, autonomous differential equations in one variable. But, as in the case of difference equations, the substitution method will become more and more tedious as the number of equations and the number of variables in the system increase. Fortunately, there exists an alternative method that obviates the dependence on the substitution method. This alternative method is called the matrix method of solution to SSDLEs and it is more general in the sense that it can be applied to SSDLEs that involve any number of equations and variables. We shall expose this method in the present section.

Suppose that we have the following general first-order, linear, autonomous SSDLEs:

$$\begin{aligned} y_1'(t) &= a_{11}y_1(t) + a_{12}y_2(t) + a_{13}y_3(t) + ... + a_{1n}y_n(t) + b_1 \\ y_2'(t) &= a_{21}y_1(t) + a_{22}y_2(t) + a_{23}y_3(t) + ... + a_{2n}y_n(t) + b_2 \\ y_3'(t) &= a_{31}y_1(t) + a_{32}y_2(t) + a_{33}y_3(t) + ... + a_{3n}y_n(t) + b_3 \,, \\ &\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ y_n'(t) &= a_{n1}y_1(t) + a_{n2}y_2(t) + a_{n3}y_3(t) + ... + a_{nn}y_n(t) + b_n \end{aligned} \tag{10.4.17}$$

where $y_i(t), i = 1,2,3,...,n$, is the i[th] variable. Notice that the system (10.4.17) is difficult for one to solve with the substitution method. Therefore, we will use the matrix method. The system (10.4.17) can be written in matrix form as

Y'(t) = **A Y**(t)+ **B**, (10.4.18)

where **Y'(t)** = $\begin{bmatrix} y_1'(t) \\ y_2'(t) \\ y_3'(t) \\ ... \\ y_n'(t) \end{bmatrix}$, **A** = $\begin{bmatrix} a_{11} & a_{12} & a_{13} & ... & a_{1n} \\ a_{21} & a_{22} & a_{23} & ... & a_{2n} \\ a_{31} & a_{32} & a_{33} & ... & a_{3n} \\ ... & ... & ... & ... & ... \\ a_{n1} & a_{n2} & a_{n3} & ... & a_{nn} \end{bmatrix}$, $\mathbf{Y_t}$ = $\begin{bmatrix} y_1(t) \\ y_2(t) \\ y_3(t) \\ ... \\ y_n(t) \end{bmatrix}$, and **B** = $\begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ ... \\ b_n \end{bmatrix}$.

Let us first determine the particular solutions to $y_i(t)$ denoted by $y_i(s)$. Notice that at the steady-state **Y'(t)** = **0**. Therefore, the system (10.4.18) in the steady-state becomes **A Y(s)** + **B** = **0** or **A Y(s)** = - **B**. Premultiplying both sides of the last equation by the inverse of **A**, $\mathbf{A^{-1}}$, if it exists, yields

Y(s) = $-\mathbf{A^{-1}}$ **B**, (10.4.19)

where **Y(s)'**=$[y_1(s) \quad y_2(s) \quad y_3(s) \quad ... \quad y_n(s)]$.

We can now attempt to determine the complementary functions. For this we shall use the homogenous version of the SSDLEs (10.4.18): **Y'(t)** = **AY(t)**. We know from our discussion in Section 10.2 that the trial solution we employed in the case of a first-order, linear, autonomous, homogenous differential equation was $y(t) = Ce^{vt}$, which implied that **Y(t)** = $e^{v_i t}$ $\mathbf{v_i}$, where $\mathbf{v_i}$ is $n \times 1$ vector of constants. The last equation implies that **Y'(t)** = $v_i e^{v_i t}$ $\mathbf{v_i}$. Substituting the last two equations into the homogenous form of the system, **Y'(t)** = **AY(t)**, we obtain $v_i e^{v_i t}$ $\mathbf{v_i}$ = **A** $e^{v_i t}$ $\mathbf{v_i}$, or v_i $\mathbf{v_i}$ = $\mathbf{Av_i}$, or [**A**- v_i **I**]$\mathbf{v_i}$ = **0**, where **I** is an $n \times n$ identity matrix. Notice that [**A**- v_i **I**]$\mathbf{v_i}$ = **0** [or **Y(t)** = $e^{v_i t}$ $\mathbf{v_i}$) is a solution to the system **Y'(t)** = **AY(t)** if v_i and $\mathbf{v_i}$ satisfy the equation [**A**- v_i **I**]$\mathbf{v_i}$ = **0**. But, the equation [**A**- v_i **I**]$\mathbf{v_i}$ = **0** implies that for **Y(t)** = $e^{v_i t}$ $\mathbf{v_i}$ to qualify for a solution (that is, for the system [**A**- v_i **I**]$\mathbf{v_i}$ = **0** to have a non-trivial solution) we must have $\mathbf{v_i} \neq \mathbf{0}$, or |**A**- v_i **I**|= **0**, which is the characteristic equation of matrix **A** or the characteristic equation associated with the system (10.4.17) and the vector $\mathbf{v_i}$ is the eigenvector associated with the eigenvalue v_i. Since matrix **A** is an $n \times n$ matrix, there will be v_i, where $i = 1,2,3,...,n$, eigenvalues, which give us n eigenvectors $\mathbf{v_i}$. Substituting this into **Y(t)** = $e^{v_i t}$ $\mathbf{v_i}$ we obtain the complementary function as

$$\mathbf{Y(c)} = e^{v_1 t}\,\mathbf{v_1} + e^{v_2 t}\,\mathbf{v_2} + e^{v_3 t}\,\mathbf{v_3} + \ldots + e^{v_n t}\,\mathbf{v_n} = \sum_{i=1}^{n} e^{v_i t}\,\mathbf{v_i}\,, \tag{10.4.20}$$

where **Y(c)'** =$[y_1(c) \quad y_2(c) \quad y_3(c) \quad ... \quad y_n(c)]$. One can show that if $e^{v_i t}$ $\mathbf{v_i}$ is a solution to the system (10.4.17), then the linear combination in equation (10.4.20) is also a solution to the same system. Let us now combine the complementary function (after multiplying it by the vector of constants **C'**=$[C_1 \quad C_2 \quad C_3 \quad ... \quad C_n]$) in equation (10.4.20) with the particular solution in equation (10.4.19) to obtain the complete solution as

$$\mathbf{Y}(t) = \mathbf{Y}(c) + \mathbf{Y}(s) = C_1 e^{v_1 t}\,\mathbf{v_1} + C_2 e^{v_2 t}\,\mathbf{v_2} + C_3 e^{v_3 t}\,\mathbf{v_3} + \ldots + C_n e^{v_n t}\,\mathbf{v_n} - \mathbf{A^{-1}}\,\mathbf{B}$$

$$= \sum_{i=1}^{n} C_i e^{v_i t}\,\mathbf{v_i} - \mathbf{A^{-1}}\,\mathbf{B}, \qquad (10.4.21)$$

which can be definitized using initial conditions. Notice that the result in equation (10.4.21) is applicable when the eigenvalues or the characteristic roots are real and distinct. When the roots are real and identical or when they are complex numbers, we need to modify the result in equation (10.4.21). But, this modification (as in the case of difference equations) is beyond the scope of this book and, therefore, we omit them.

As an application of the matrix method, consider the problem we solved through the substitution method in the last section. The SSDLEs in this example is in equation (10.4.16): $y'(t) = -2y(t) - 2x(t) + 80$, and $x'(t) = 0 \times y(t) - x(t) + 20$,which can be written in matrix form as

$$\mathbf{Y'(t)} = \mathbf{A}\ \mathbf{Y(t)} + \mathbf{B}, \qquad (10.4.22)$$

where $\mathbf{Y'(t)} = \begin{bmatrix} y'(t) \\ x'(t) \end{bmatrix}$, $\mathbf{A} = \begin{bmatrix} -2 & -2 \\ 0 & -1 \end{bmatrix}$, $\mathbf{Y(t)} = \begin{bmatrix} y(t) \\ x(t) \end{bmatrix}$, and $\mathbf{B} = \begin{bmatrix} 80 \\ 20 \end{bmatrix}$. Let us first determine the particular solutions. In the steady-state the variables are such that $dy(t)/dt = y'(t) = dx(t)/dt = x'(t) = 0$. Thus, following equation (10.4.19), we can write the steady-state solutions as

$$\mathbf{Y(s)} = -\mathbf{A^{-1}}\,\mathbf{B}, \qquad (10.4.23)$$

where $\mathbf{Y(s)} = \begin{bmatrix} y(s) \\ x(s) \end{bmatrix}$. The inverse of the matrix **A** is $\mathbf{A^{-1}} = \begin{bmatrix} -1/2 & 1 \\ 0 & -1 \end{bmatrix}$. Therefore, applying equation (10.4.23), we obtain the particular solutions $\mathbf{Y}(s) = \begin{bmatrix} y(s) \\ x(s) \end{bmatrix} = -\mathbf{A^{-1}}\ \mathbf{B} = -\begin{bmatrix} -1/2 & 1 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} 80 \\ 20 \end{bmatrix} = \begin{bmatrix} 20 \\ 20 \end{bmatrix}$. This means that $y(s) = 20$ *and* $x(s) = 20$. Notice that these were exactly the particular solutions we obtained when we applied the substitution method to the same problem in the last section.

We shall now attempt to find the complementary functions using equation (10.4.20). But, to apply this equation, we first need to find the eigenvalues and eigenvectors associated with the coefficient matrix **A**. The eigenvalues can be obtained as $v_1 = -2$ *and* $v_2 = -1$. Notice that these were the same roots as those we obtained in the last section. The eigenvectors of matrix **A** or the eigenvectors associated with eigenvalues $v_1 = -2$ *and* $v_2 = -1$ are $\mathbf{v_1'} = [1 \quad 0]$ and $\mathbf{v_2'} = [1 \quad -0.5]$. Therefore, applying equation (10.2.20), we obtain the complementary functions $\mathbf{Y(c)} = \begin{bmatrix} y(c) \\ x(c) \end{bmatrix} = e^{v_1 t}\,\mathbf{v_1} + e^{v_2 t}\,\mathbf{v_2} = e^{-2t}\begin{bmatrix} 1 \\ 0 \end{bmatrix} + e^{-t}\begin{bmatrix} 1 \\ -0.5 \end{bmatrix} = \begin{bmatrix} e^{-2t} & e^{-t} \\ 0 & -0.5e^{-t} \end{bmatrix}$. We can multiply the last result with the vector of constants **C'** to obtain $\mathbf{Y(c)} = \begin{bmatrix} y(c) \\ x(c) \end{bmatrix} = \begin{bmatrix} e^{-2t} & e^{-t} \\ 0 & -0.5e^{-t} \end{bmatrix}\begin{bmatrix} C_1 \\ C_2 \end{bmatrix} = \begin{bmatrix} C_1 e^{-2t} + C_2 e^{-t} \\ -0.5C_2 e^{-t} \end{bmatrix}$. We now add the particular solutions to the complementary functions, as required by equation (10.4.21), to obtain the complete solution $\mathbf{Y(t)} = \begin{bmatrix} y(t) \\ x(t) \end{bmatrix} = \mathbf{Y(c)} + \mathbf{Y(s)} = \begin{bmatrix} y(c) \\ x(c) \end{bmatrix} + \begin{bmatrix} y(s) \\ x(s) \end{bmatrix} = \begin{bmatrix} e^{-2t} & e^{-t} \\ 0 & -0.5e^{-t} \end{bmatrix}\begin{bmatrix} C_1 \\ C_2 \end{bmatrix} + \begin{bmatrix} 20 \\ 20 \end{bmatrix} =$ $\begin{bmatrix} C_1 e^{-2t} + C_2 e^{-t} \\ -0.5C_2 e^{-t} \end{bmatrix} + \begin{bmatrix} 20 \\ 20 \end{bmatrix}$, or $y(t) = y(c) + y(s) = C_1 e^{-2t} + C_2 e^{-t} + 20$ and $x(t) = x(c) + x(s) = -0.5C_2 e^{-t} + 20$. Definitizing these solutions using initial conditions $y(0) = 20$ *and* $x(0) = 10$ when $t = 0$ to yield $C_1 = -20\,and\,C_2 = 20$. Then the complete and definite solutions become $y(t) = -20e^{-2t} + 20e^{-t} + 20$ and $x(t) = -10e^{-t} + 20$. The reader must have noticed that these were exactly the same solutions as those we obtained to the SSDLEs and for the same initial conditions in the last section through the application of the substitution method.

10.4.4 SSDLEs: convergence

In Sections 10.2 and 10.3 we discussed the convergence of the solution to first-order and higher-order single differential equations, respectively. We found in both these discussions that the convergence of the solutions to these equations depended solely on the sign of the characteristic roots associated with the equations. Specifically, our result was that the solution path of the variable would converge to (diverge from) its steady-state value if the signs of the roots were negative (positive). This is valid irrespective of the order of the differential equation. But, as we observed in Section 10.3.6, obtaining the roots of higher-order differential equations is a difficult task and, therefore, a quantitative analysis of convergence is often difficult. Even in this case we can carry out a qualitative analysis using Routh-Hurwitz theorem outlined in Section 10.3.6.

One might wonder how we can get information on convergence in the case of linear, autonomous SSDLEs with which we were concerned so far in the present section. A close inspection of equation (10.4.20) or (10.4.21) would suggest the clue. Notice that the movements of $y(t)$ *and* $x(t)$ in equation (10.4.21) are still dependent upon the exponent v_i . We know that these exponents are the characteristic roots or the concerned SSDLEs.

The above discussion points to the fact that, as in the case of single linear differential equations, convergence of the time paths of variables in SSDLEs is also dependent upon the signs of the characteristic roots of the system. If the signs of all the associated roots (real part in the case of complex roots) are negative (positive), then the time paths of the variables of the system will converge to (diverge from) the steady-state value. These were the reasons why the time paths of the variables x *and* y in the solutions in the example in Section 10.4.2 converged to their respective and identical steady-state value as exemplified by the graphs in Figure 10.4.1.

10.4.5 Phase plane analysis of SSDLEs

We exposed the phase diagram analyses of single linear and nonlinear differential equations in Sections 10.2.4 and 10.2.6, respectively. In these sections our aim was to draw qualitative inferences about the time paths of the variables of the equations even without solving them explicitly. Our main inferences were that the slope of the phase line at its intersection point with the horizontal axis determined whether the underlying variable of the differential equation converged to, or diverged from, or cycled between the steady-state value(s). If the slope of the phase line is negative (positive) or infinite, then the dependent variable will converge to (diverge from) or cycles between steady-state value(s).

As in the case of the solution to the system (10.4.16), if we could solve SSDLEs explicitly we could judge whether the time paths of the variables of the system would converge to or diverge from the steady-state using the signs of the associated characteristic roots. But, some SSDLEs, whether linear or nonlinear, are difficult to be solved explicitly. Even in this event we can conduct a qualitative analysis of the nature of the time paths of the variables and this analysis is called phase plane analysis of SSDLEs, which is the topic of the present section.

Let us illustrate the phase plane analysis with the same example of the two-variable, two-equation SSDLEs (10.4.16): $y'(t) = -2y(t) - 2x(t) + 80$, and $x'(t) = 0 \times y(t) - x(t) + 20$. If $x'(t) = 0$ and $y'(t) = 0$, then the variables $x(t)$ and $y(t)$ are at their steady-states. Assume for the moment that $y'(t) = x'(t) = 0$. This helps us write the two equations in the system (10.4.16) as $x'(t) = 0 \Rightarrow f[x(t), y(t)] = 0 \times y(t) - x(t) + 20 = 0$ and $y'(t) = 0 \Rightarrow g[x(t), y(t)] = -2y(t) - 2x(t) + 80 = 0$, which can be solved for $y(t)$ to obtain

$$x'(t) = 0 \Rightarrow x(t) = 20 \quad \text{and} \quad y'(t) = 0 \Rightarrow y(t) = 40 - x(t), \tag{10.4.24}$$

which are plotted in Panel (A) of Figure 10.4.2 and denoted as $x'(t) = 0$ and $y'(t) = 0$ lines. As we saw in the case of difference equations, these lines are called demarcation lines in the case of differential equations.

Notice few points about the demarcations lines in Panel (A) of Figure 10.4.2. The first is that both lines are straight lines because the associated first derivatives of $y(t)$ with respect to $x(t)$ in the system (10.4.16) are both constants; that is, $dy(t)/dx(t)\big|_{x'(t)=0} = \infty$ and $dy(t)/dx(t)\big|_{y'(t)=0} = -1$. The second point is that these lines have different slopes; that is, $dy(t)/dx(t)\big|_{x'(t)=0} = \infty > 0$ and $dy(t)/dx(t)\big|_{y'(t)=0} = -1 < 0$ implying that the line representing $x'(t) = 0$ is parallel to the vertical axis and the line representing $y'(t) = 0$ is lying downward from left to right. Moreover, since they are linear lines and have different slopes, the demarcation lines will cross each other at only one point (s in the panel) in the [$x(t), y(t)$] plane.

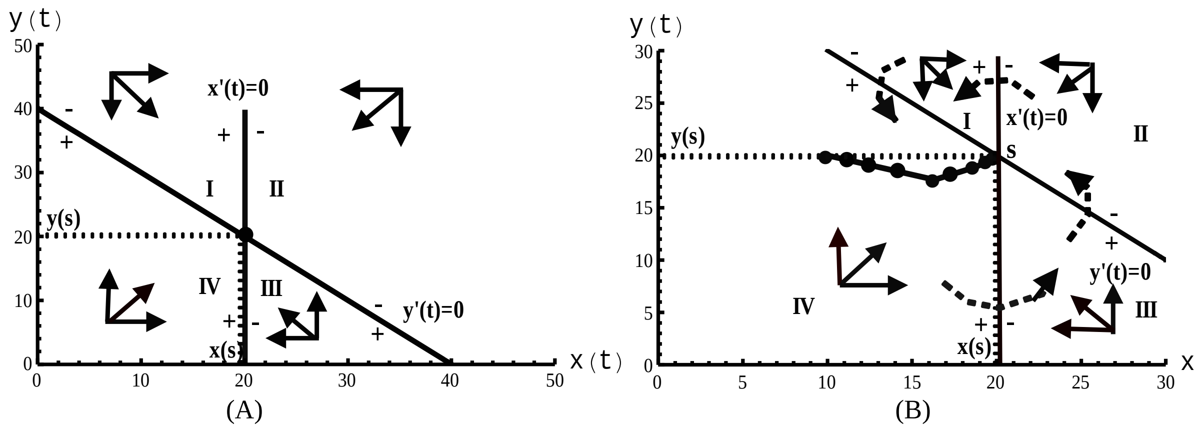


Figure 10.4.2

Let us now see the behavior of the variables $x(t)$ *and* $y(t)$ in the phase plane. This can be understood by inspecting the partial derivatives of the system (10.4.16) with respect to x and y. If $\partial x'(t)/\partial x(t) > 0$ $[\partial y'(t)/\partial y(t) > 0]$, an increase in $x(t)$ $[y(t)]$ increases $x'(t)[y'(t)]$ and, therefore, $x(t)[y(t)]$ increases eventually. If $\partial x'(t)/\partial x(t) < 0$ $[\partial y'(t)/\partial y(t) < 0]$, an increase in $x(t)$ $[y(t)]$ decreases $x'(t)[y'(t)]$ and, therefore, $x(t)[y(t)]$ decreases eventually. These imply that if the change in $x'(t)[y'(t)]$ is positive when $x(t)[y(t)]$ changes, then $x(t)$ $[y(t)]$ will increase. The opposite will happen if the change in $x'(t)[y'(t)]$ is negative when $x(t)[y(t)]$ changes.

In our example, we obtain from the system that $\partial x'(t)/\partial x(t) = -1 < 0$. Since the partial derivative is negative, $x'(t)$ must decrease (increase) as $x(t)$ increases (decreases) and, consequently $x(t)$ decreases (increases). Therefore, on the RHS of the demarcation line $x'(t) = 0$ we have a decrease in $x(t)$. But, if $x(t)$ decreases on the RHS of the demarcation line $x'(t) = 0$, it must increase on the LHS of the line. In short, we have a decreasing $x(t)$ and an increasing $x(t)$ on the RHS and LHS, respectively, of the demarcation line $x'(t) = 0$. We denote these effects on $x(t)$ by minus and plus signs (or by leftward and rightward pointing arrows), respectively, in Panel (A) of Figure 10.4.2. Similarly, we have $\partial y'(t)/\partial y(t) = -2 < 0$. Since the partial derivative is negative, $y'(t)$ must decrease (increase) as $y(t)$ increases (decreases) and, consequently $y(t)$ decreases (increases). Therefore, above the demarcation line $y'(t) = 0$ we have a decrease in $y(t)$. But, if $y(t)$ decreases above the demarcation line $y'(t) = 0$, it must increase below the line. In short, we have a decreasing $y(t)$ and an increasing $y(t)$ above and blow, respectively, of the demarcation line $y'(t) = 0$. We denote in the panel these effects by minus and plus signs (or by downward and upward pointing arrows), respectively. What all these mean are that both $x(t)$ *and* $y(t)$ increase in phase **IV**, both $x(t)$ *and* $y(t)$ decrease in phase **II**, $y(t)$ decreases and $x(t)$ increases in phase **I**, and $y(t)$ increases and $x(t)$ decreases in phase **III** in the panel.

It is now clear that, no matter where $x(t)$ *and* $y(t)$ begin with their initial values, they will eventually converge to the point s , the point that corresponds to their common and identical steady-state value. But, how do we find the values that correspond to the point s ? Notice that the point s is common to both demarcation lines and these demarcation lines, as mentioned above, are defined by the two equations in the system (10.4.24). This implies that the values that correspond to the point s can be found by solving the two simultaneous equations in the system (10.4.24). Therefore, the solution is $x(t) = x(s) = y(t) = y(s) = 20$.

We know now that the intertemporal behavior of the variables in SSDLEs, as in the case of difference equations, can be determined qualitatively through the phase plane analysis. In our specific example above, we found that both variables converged to the common steady-state value. But, this was the same conclusion we obtained with Figure 10.4.1, which was based on the explicit solution of the same SSDLEs. The advantage of the phase plane analysis, therefore, lies in the determination of the behavior of variables in a SSDLEs that is difficult to be solved explicitly.

As in the case of difference equations, we may need to know how the variables move if they happen, with initial values, somewhere in any one of the phases in Panel (A) of Figure 10.4.2. If we didn't solve the system explicitly, we can only conjecture how the variables move. They may start with initial values at a point in any one of the phases and may take routes including those represented by the dashed arrows in Panel (B) of the same figure. We know from the last chapter that these arrows are called trajectories or streamlines. If we have already solved the system explicitly then we can use these solutions to construct a scatter diagram of the variables and, then, superimpose the scatter diagram on the graphs in Panel (A) of the figure. The result will give us the exact path of the movement or exact trajectory of the variables to the steady-state as represented by the path of thick dots in Figure in Panel (B); or movement away from it. Notice that this path is also a trajectory. Notice also that one could obtain a variety of phase planes and, thereby, a variety of equilibria depending upon the forms of demarcation lines and the values of the associated derivatives.

10.4.6 Simultaneous systems of nonlinear differential equations

We considered single, nonlinear, autonomous differential equations in Section 10.2.6. We discussed there how to linearize nonlinear differential equations and, then, solve them. However, our discussion there was mainly centered on deriving the qualitative information about the convergence of the variable of the equation. This was due to the fact that it was difficult to solve many of the nonlinear differential equations explicitly. Our qualitative analysis suggested that the sign of the first-order derivative of the phase line evaluated at the steady-state value determined the convergence of the variable. If the first-order derivative of the phase line evaluated at the steady-state value was negative (positive), then we observed that the variable of the equation would converge to (diverge from) its steady-state.

We saw in the last section how one could derive qualitative information about the intertemporal behavior of variables involved in a SSDLEs. But, we know that it is much better if we can solve the system explicitly, which obviates the need for qualitative analysis. This is often achieved by linearizing **systems of nonlinear differential equations** (hereafter, **SSNDLEs**). Let us now consider SSNDLEs and see how we can solve them and, thereby, obtain information about the convergence of their variables.

As an illustration, consider the SSNDLEs

$$y'(t) = f[x(t), y(t)] \quad \text{and} \quad x'(t) = g[x(t), y(t)], \tag{10.4.25}$$

where $f[x(t), y(t)]$ and $g[x(t), y(t)]$ are assumed to be nonlinear. We know that in the steady-state, if it exists, we have $x'(t) = y'(t) = 0$ and the steady-state values of the two variables are denoted by $y(s)$ *and* $x(s)$. Let us now linearize the system (10.4.25) around $y(s)$ *and* $x(s)$ using Taylor series introduced in Section 3.6 to obtain

$$y'(t) = f[x(s), y(s)] + \frac{\partial f[x(s), y(s)]}{\partial x(t)}[x(t) - x(s)] + \frac{\partial f(x_s, y_s)}{\partial y(t)}[y(t) - y(s)] \text{ and}$$
$$x'(t) = g[x(s), y(s)] + \frac{\partial g[x(s), y(s)]}{\partial x(t)}[x(t) - x(s)] + \frac{\partial g(x_s, y_s)}{\partial y(t)}[y(t) - y(s)], \tag{10.4.26}$$

where we have omitted the remainder terms for convenience. But, we know that at the steady-state $y'(t) = x'(t) = 0$, which implies that $f[x(s), y(s)] = 0$ *and* $g[x(s), y(s)] = 0$. Therefore, the system (10.4.26) can be written as

$$y'(t) = \frac{\partial f[x(s), y(s)]}{\partial x(t)}[x(t) - x(s)] + \frac{\partial f(x_s, y_s)}{\partial y(t)}[y(t) - y(s)] \text{ and}$$
$$x'(t) = \frac{\partial g[x(s), y(s)]}{\partial x(t)}[x(t) - x(s)] + \frac{\partial g(x_s, y_s)}{\partial y(t)}[y(t) - y(s)]. \tag{10.4.27}$$

Notice that the system (10.4.27) can also be written as

$y'(t)=\frac{\partial f[x(s),y(s)]}{\partial x(t)}x(t)+\frac{\partial f(x_s,y_s)}{\partial y(t)}y(t)+b_{f\,y}$ and $x'(t)=\frac{\partial g[x(s),y(s)]}{\partial x(t)}x(t)+\frac{\partial g(x_s,y_s)}{\partial y(t)}y(t)+b_{g\,x}$,

where $b_{f\,y}=-\left[\frac{\partial f[x(s),y(s)]}{\partial x(t)}x(s)+\frac{\partial f(x_s,y_s)}{\partial y(t)}y(s)\right]$ and $b_{g\,x}=-\left[\frac{\partial g[x(s),y(s)]}{\partial x(t)}x(s)+\frac{\partial g(x_s,y_s)}{\partial y(t)}y(s)\right]$. Let us now define $a_{11}=\partial f[x(s),y(s)]/\partial x(t);$ $a_{12}=\partial f[x(s),y(s)]/\partial y(t);$ $a_{21}=\partial g[x(s),y(s)]/\partial x(t);$ and $a_{22}=\partial g[x(s),y(s)]/\partial y(t)$. Using these definitions, the last system can be written as

$y'(t)=a_{11}x(t)+a_{12}y(t)+b_{f\,y}$ and $x'(t)=a_{21}x(t)+a_{22}y(t)+b_{g\,x}$, (10.4.28)

which can be written in matrix form as

$\mathbf{Y'(t)} = \mathbf{A_{fg}}\ \mathbf{Y_t} + \mathbf{B_{fg}}$, (10.4.29)

where $\mathbf{Y'(t)}=\begin{bmatrix} y'(t) \\ x'(t) \end{bmatrix}$, $\mathbf{A_{fg}}=\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}=\begin{bmatrix} \frac{\partial f}{\partial x(t)} & \frac{\partial f}{\partial y(t)} \\ \frac{\partial g}{\partial x(t)} & \frac{\partial g}{\partial y(t)} \end{bmatrix}$, $\mathbf{Y_t}=\begin{bmatrix} x_t \\ y_t \end{bmatrix}$ and $\mathbf{B_{fg}}=\begin{bmatrix} b_{f\,y} \\ b_{g\,x} \end{bmatrix}$. Equation (10.4.29) is similar to equation (10.4.18) with solution in equation (10.4.21). Therefore, applying equation (10.4.21), we obtain the solution to the system (10.4.28) or (10.4.29) as

$\mathbf{Y(t)} = \mathbf{Y(c)} + \mathbf{Y(s)} = \sum_{i=1}^{n} C_i e^{v_i\,t}\,\mathbf{v_i} - \mathbf{A_{fg}}^{-1}\,\mathbf{B_{fg}}$, (10.4.30)

where C is the constant to be definitized, $\mathbf{v_i}$ is the characteristic vector associated with the characteristic root v_i of the coefficient Jacobian matrix $\mathbf{A_{fg}}$, $\mathbf{Y(c)}=\sum_{i=1}^{n} C_i e^{v_i\,t}\,\mathbf{v_i}$, and $\mathbf{Y(s)} = -\mathbf{A_{fg}}^{-1}\,\mathbf{B_{fg}}$. Notice that the solution (10.4.30) is the solution to the linearized system (10.4.28) of the nonlinear system (10.4.25). It can be shown that the intertemporal behaviors of the solutions to both systems are similar.

10.4.7 Application examples

Example 1: Consider a variant of the Walrasian price-adjustment model we solved in example 3 in 10.3.7. Suppose that the quantity demanded, $q_d(t)$, and the quantity supplied, $q_s(t)$, at time t are given by $q_d(t)=30-4p(t)$ and $q_s(t)=2F(t)$, respectively, where $p(t)$ *and* $F(t)$ denote per unit price and the per unit profit, respectively. Also suppose that the rates of changes of per unit price and the per unit profit are given by $dp(t)/dt=0.5[q_d(t)-q_s(t)]$ and $dF(t)/dt=0.5[p(t)-5]$, respectively. All values are in dollars. Construct a phase plane for the model and determine the movement of the trajectory of price and the per unit profit.

Solution: Let us first set up the system of differential equations for the model. Substituting the demand and supply functions into the rate of change of price and simplifying yield $p'(t)=-2p(t)-F(t)+15$. The rate of change of per unit profit can be written as $F'(t)=0.5p(t)+0\times F(t)-2.5$. Therefore, the SSDLEs for the model can be written as $p'(t)=-2p(t)-F(t)+15$ and $F'(t)=0.5p(t)+0\times F(t)-2.5$, which can be written in matrix form as $\mathbf{Y'(t)} = \mathbf{AY(t)}+\mathbf{B}$, where $\mathbf{Y'}$(t)$=\begin{bmatrix} p'(t) \\ F'(t) \end{bmatrix}$,$\mathbf{A}=\begin{bmatrix} \frac{\partial p'(t)}{\partial p(t)} & \frac{\partial p'(t)}{\partial F(t)} \\ \frac{\partial F'(t)}{\partial p(t)} & \frac{\partial F'(t)}{\partial F(t)} \end{bmatrix}=\begin{bmatrix} -2 & -1 \\ 0.5 & 0 \end{bmatrix}$,$\mathbf{Y(t)}=\begin{bmatrix} p(t) \\ F(t) \end{bmatrix}$, and $\mathbf{B}=\begin{bmatrix} 15 \\ -2.5 \end{bmatrix}$.

To solve the present model we shall use the matrix method exposed in Section 10.4.3. We can first determine the particular solutions. Applying equation (10.4.19), the particular solutions can be found as **Y(s)** = $\begin{bmatrix} p(s) \\ F(s) \end{bmatrix}$ = **-A⁻¹ B** = $-\begin{bmatrix} 0 & 2 \\ -1 & -4 \end{bmatrix}\begin{bmatrix} 15 \\ -2.5 \end{bmatrix} = \begin{bmatrix} 5 \\ 5 \end{bmatrix}$, or $p(s) = 5$ *and* $F(s) = 5$.

Let us now determine the complementary functions. For this we can apply equation (10.4.20): **Y(c)** = $e^{v_1 t}$ $\mathbf{v_1}$ + $e^{v_2 t}$ $\mathbf{v_2}$ + $e^{v_3 t}$ $\mathbf{v_3}$ +…+ $e^{v_n t}$ $\mathbf{v_n}$ = $\sum_{i=1}^{n} e^{v_i t}$ $\mathbf{v_i}$. But, to apply this equation we need to find the characteristic roots (v_i) and the characteristic vectors ($\mathbf{v_i}$) associated with the coefficient matrix **A**. They can be obtained as $v_1 = -1.707$ *and* $v_2 = -0.2929$, and $\mathbf{v_1}'$ = $[-0.9596 \quad 0.5054]$ and $\mathbf{v_2}'$ = $[0.2811 \quad -0.8629]$, respectively. Therefore, the complementary functions can be written, applying equation 10.4.20), as **Y(c)**= $e^{v_1 t}$ $\mathbf{v_1}$ + $e^{v_2 t}$ $\mathbf{v_2}$ = $e^{-1.707t}\begin{bmatrix} -0.9596 \\ 0.5054 \end{bmatrix} + e^{-0.2929t}\begin{bmatrix} 0.2811 \\ -0.8629 \end{bmatrix} = \begin{bmatrix} -0.9596e^{-1.707t} + 0.2811e^{-0.2929t} \\ 0.5054e^{-1.707t} - 0.8629e^{-0.2929t} \end{bmatrix}$. We can now combine the particular solutions with the complementary functions, after multiplying them with the constant **C'** =$[C_1 \quad C_2]$, to obtain the complete solutions. Thus, applying equation (10.4.21), we obtain the complete solutions **Y(t)** = **Y(c)** + **Y(s)** $= C_1 e^{v_1 t}$ $\mathbf{v_1}$ $+ C_2 e^{v_2 t}$ $\mathbf{v_2}$ **-A⁻¹ B** = $\begin{bmatrix} -0.9596\, C_1 e^{-1.707\,t} + 0.2811\, C_2 e^{-0.2929\,t} \\ 0.5054\, C_1 e^{-1.707\,t} - 0.8629\, C_2 e^{-0.2929\,t} \end{bmatrix} + \begin{bmatrix} 5 \\ 5 \end{bmatrix}$. Assume, for convenience, that $C_1 = C_2 = -1$. Then the complete and definite solutions can be written as **Y(t)** = $\begin{bmatrix} p(t) \\ F(t) \end{bmatrix}$= $\begin{bmatrix} 0.9596\,e^{-1.707t} - 0.2811e^{-0.2929\,t} \\ -0.5054\,e^{-1.707\,t}\,0.8629\,e^{-0.2929\,t} \end{bmatrix} + \begin{bmatrix} 5 \\ 5 \end{bmatrix}$, or $p(t) = 0.9596\,e^{-1.707\,t} - 0.2811\,e^{-0.2929\,t} + 5$, and $F(t) = -0.5054\,e^{-1.707\,t} + 0.8629\,e^{-0.2929\,t} + 5$. The graphs of these two solutions are illustrated in Panel (A) of Figure 10.4.3, which show that the per unit price and the per unit profit converge to their identical steady-state values $5.

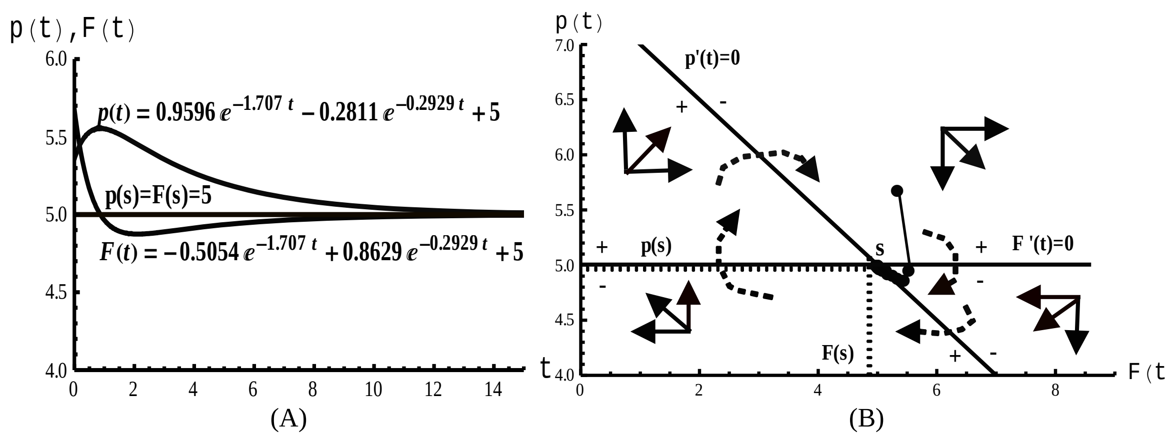


Figure 10.4.3

Let us now determine the trajectory, in the associated phase plane, that represents the movements of price and per unit profit. For this we proceed exactly as we did in Section 10.4.5. We know that, when price and the per unit profit are in their steady -states, $p'(t) = F'(t) = 0$. Given this result, we can solve the differential equations for $p(t)$ as $p(t) = 7.5 - 0.5F(t)$ and $p(t) = 5$, which are plotted to obtain the demarcation lines (straight lines) in Panel (B) of Figure 10.4.3. Notice that $dp'(t)/dp(t) = -2 < 0$ and $dF'(t)/dF(t) = 0$. Since $dp'(t)/dp(t) = -2 < 0$, $p(t)$ decreases (increases) on the right (left) of the demarcation line $p'(t) = 0$. Moreover, we know from the differential equation $F'(t) = 0.5p(t) + 0 \times F(t) - 2.5$ that $F'(t) > 0$ $[F'(t) < 0]$ and, thus, $F(t)$ increases (decreases) eventually if $p(t) > 5$ $[p(t) < 5]$. Therefore, $F(t)$ increases (decreases) above (below) the demarcation line $F'(t) = 0$. These increases and decreases in $p(t)$ *and* $F(t)$ are shown by the positive and negative signs (or by upward and downward, and rightward and leftward pointing arrows), respectively. Notice that we obtain $dp(t)/dF(t)\big|_{p'(t)=0} = -0.5$ and $dp(t)/dF(t)\big|_{F'(t)=0} = 0$. The last two equations imply that the demarcation lines $p'(t) = 0$ *and* $F'(t) = 0$ decrease from left to write and remain parallel to the horizontal axis, respectively. They also imply that the demarcation lines will cross each other only at one point, and that point is the steady-state point s at which $p(s) = F(s) = 5$. The last step is to plot the scatter diagram of $p(t)$ *and* $F(t)$ from the solutions and superimpose it on the graph of the demarcation lines which yield the required trajectory (the curve with heavy dots) as shown in Panel (B) of the same figure. As can be seen from the panel, the trajectory tends to the equilibrium point.

Example 2: Suppose that a continuous time version of the expectations- augmented Phillips curve is given by the SSDLEs $dp^e(t)/dt = -p^e(t) + 3u(t) - 5$ and $du(t)dt = -p^e(t) - u(t) + 5$, where $p^e(t)$ *and* $u(t)$ denote the expected rate of inflation and the unemployment rate, respectively. Construct a phase plane for the model and determine the movement of the trajectory of the expected rate of inflation and the unemployment rate.

Solution: Notice that the above SSDLEs is identical with the system (10.4.1) for $p^e = y, u = x, a_{11} = -1, a_{12} = 3, a_{21} = -1, a_{22} = -1,\ b_p = b_y = -5,\ \textit{and}\ b_u = b_x = 5$. We shall apply the substitution method exposed in Section 10.4.2 to solve this problem. Let us first determine the particular solutions using equations (10.4.2) and (10.4.3): $p^e(s) = [(a_{12}b_u - a_{22}b_p)/(a_{11}a_{22} - a_{12}a_{21})] = [(3\times 5 - (-1\times -5)]/[(-1\times -1-(3\times -1)] = 2.5$ and $u(s) = [(a_{21}b_p - a_{11}b_u)/(a_{11}a_{22} - a_{12}a_{21})] = [-1\times -5 - (-1\times 5)]/[(-1\times -1 - (3\times -1)] = 2.5$. Therefore, the steady-state values of expected rate of inflation and unemployment rate are the same 2.5.

Let us now attempt to find the complementary functions. For this we shall use the homogenous versions of the system: $dp^e(t)/dt = -p^e(t) + 3u(t)$ and $du(t)/dt = -p^e(t) - u(t)$. Differentiation of the first equation with respect to time t yields $d^2p^e(t)/dt^2 = -dp^e(t)/dt + 3du(t)/dt$. We can substitute $du(t)/dt = -p^e(t) - u(t)$ into the last equation to obtain $d^2p^e(t)/dt^2 = -dp^e(t)/dt - 3p^e(t) - 3u(t)$. We may now solve $dp^e(t)/dt = -p^e(t) + 3u(t)$ for $u(t)$ and substitute the result into $d^2p^e(t)/dt^2 = -dp^e(t)/dt - 3p^e(t) - 3u(t)$ to get $d^2p^e(t)/dt^2 = -2dp^e(t)/dt - 2p^e(t)$. Notice that the last equation is identical with equation (10.4.6) for $p^e = y, a_1 = (a_{11} + a_{22}) = -2,\ \textit{and}\ a_2 = (a_{12}a_{21} - a_{11}a_{22}) = -2$. Using the trial solution $y(t) = Ce^{vt}$ with $dy(t)/dt = Cve^{vt}$ and $d^2y(t)/dt^2 = Cv^2e^{vt}$, and substituting them into the differential equation $d^2p^e(t)/dt^2 = -2dp^e(t)/dt - 2p(t)$ yield $Cv^2e^{vt} = -2Cve^{vt} - 2Ce^{vt}$; or $Cv^2e^{vt} + 2Cve^{vt} + 2Ce^{vt} = 0$; or $Ce^{vt}[v^2 + 2v + 2] = 0$. We can obtain from the last equation the characteristic roots $v_1, v_2 = u \pm qi$, where $u = -1$ *and* $q = 1$. Then, using equations (10.4.11) and (10.4.12), the complementary functions for $p^e(t)$ *and* $u(t)$ can be written as $p^e(c) = e^{ut}[C_1\cos(qt) + C_2\sin(qt)],\ u(c) = e^{ut}\{[C_1(u - a_{11}) + C_2q]/a_{12}\}\cos(qt) + e^{ut}\{[C_2(u - a_{11}) - C_1q]/a_{12}\}\sin(qt)$, respectively. Substituting the values $a_{11} = -1, a_{12} = 3, u = -1,\ \textit{and}\ q = 1$, the complementary functions can be written as $p^e(c) = e^{-t}[C_1\cos(t) + C_2\sin(t)]$ and $u(c) = e^{-t}[(-2C_1 + C_2)/3]\cos(t) + e^{-t}\{[(-2C_2 - C_1)/3]\}\sin(t)$. We can now combine the complementary functions with the particular solutions, or apply equation (10.4.15), to obtain the complete solutions $p^e(t) = e^{-t}[C_1\cos(t) + C_2\sin(t)] + 2.5$ and $u(c) = e^{-t}[(-2C_1 + C_2)/3]\cos(t) + e^{-t}\{[(-2C_2 - C_1)/3]\}\sin(t) + 2.5$. Suppose, for simplicity, that $C_1 = C_2 = 1$. Substitution of these values into the complete solutions yields the complete and definite solutions $p^e(t) = e^{-t}[\cos(t) + \sin(t)] + 2.5$ and $u(t) = e^{-t}[-0.33\cos(t) - \sin(t)] + 2.5$, which are plotted in Panel (A) of Figure 10.4.4. These plots show that both the expected rate of inflation and the unemployment rate converge to their common steady-state value (2.5).

Let us now determine the trajectory, in the associated phase plane, that represents the movements of the variables. For this we follow the exposition in Section 10.4.5. We know that, when the rate of expected inflation and the unemployment are in their steady-states, $dp^e(t)/dt = du(t)/dt = 0$. Given this, we can solve the differential equations for $p^e(t)$ as $p^e(t) = -5 + 3u(t)$ and $p^e(t) = 5 - u(t)$, which are plotted to obtain the demarcation lines (straight lines) in Panel (B) of Figure 10.4.4. Notice that $dp^e{}'(t)/dp^e(t) = -1 < 0$ and $du'(t)/du(t) = -1 < 0$. Since both $dp^e{}'(t)/dp^e(t)$ and $du'(t)/du(t)$ are negative, $p^e(t)$ [$u(t)$] decreases on the right (above) and increases on the left (below) of the demarcation line $dp^e(t)/dt = p^e{}'(t) = 0$ [$du(t)/dt = u'(t) = 0$]. These increase and decrease, as before, are shown by the positive and negative signs (or by upward and rightward, and downward and leftward pointing arrows), respectively. Moreover, we obtain that $dp^e(t)/du(t)\big|_{p^e{}'(t)=0} = 3$ and $dp^e(t)/du(t)\big|_{u'(t)=0} = -1$. The last two equations imply that the demarcation lines $dp^e(t)/dt = 0$ *and* $du(t)/dt = 0$ increase from left to write and decrease from left to right, respectively. They also imply that the demarcation lines will cross each other only at one point, and that point is the steady-state or equilibrium point s with coordinates $p^e(s) = u(s) = 2.5$. The last step, as before, is to plot the scatter diagram of $p^e(t)$ *and* $u(t)$ from the solutions and superimpose it on the graph of the demarcation lines which yield the required trajectory (the curve with heavy dots) as shown in Panel (B) of the same figure. As can be seen from the panel, the trajectory tends to the equilibrium point.

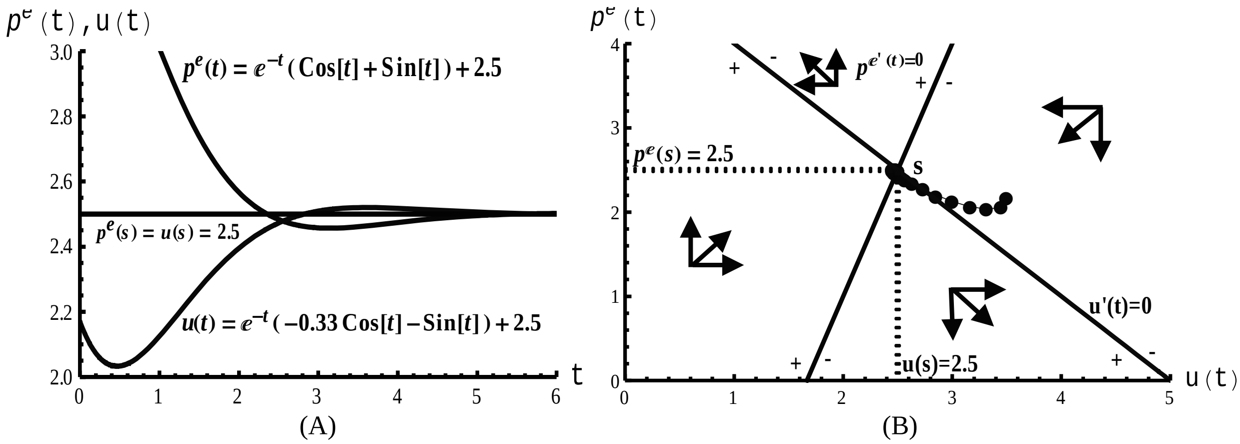


Figure 10.4.4

Example 3: Suppose that the total output $Y(t)$ produced at time t in an economy using physical capital [$K(t)$], human capital [$H(t)$], labor [$L(t)$] and technology [$A(t)$] is given by the Cobb-Douglas production function $Y(t) = K(t)^{\alpha} H(t)^{\beta} [A(t)L(t)]^{1-\alpha-\beta}$, where α *and* β denote the output elasticities of physical and human capital, respectively; and $\alpha > 0, \beta > 0,$ *and* $\alpha + \beta < 1$. Total saving (s) is assumed to be constant and is invested in physical and human capital with constant fractions s_K *and* s_H respectively. This implies that physical and human capitals grow over time as $dK(t)/dt = K'(t) = s_K Y(t)$ and $dH(t)/dt = H'(t) = s_H Y(t)$. Labor and technology are assumed to grow over time at constant rates n *and* g, respectively: $L'(t) = n$ and $A'(t) = g$. Lastly, assume that $y(t) = Y(t)/[A(t)L(t)], k(t) = K(t)/[A(t)L(t)],$ *and* $h(t) = H(t)/[A(t)L(t)]$, where $[A(t)L(t)]$ is referred to as effective labor. Derive the SSDLEs in $k(t)$ *and* $h(t)$ and show that they converge to their steady-state values $k(s)$ *and* $h(s)$.

Solution: This is an example of the **continuous time version** of Mankiw-Romer-Weil model of economic growth. Given the above definitions, we can derive, following a procedure similar to the one adopted in example 10 in Section 10.2.7, that $y(t) = k(t)^{\alpha} h(t)^{\beta}$, $k'(t) = s_K y(t) - (n+g)k(t)$ and $h'(t) = s_H y(t) - (n+g)h(t)$, which denote the output, the change in physical capital, and the change in human capital per effective labor, respectively. Substituting $y(t) = k(t)^{\alpha} h(t)^{\beta}$ into the last two equations yields $k'(t) = s_K k(t)^{\alpha} h(t)^{\beta} - (n+g)k(t)$ and $h'(t) = s_H k(t)^{\alpha} h(t)^{\beta} - (n+g)h(t)$. The last two equations represent the required SSDLEs. Notice that these differential equations are nonlinear. The intertemporal behavior of the variables of these equations can be inferred either through linearization or through phase plane analysis. As it is much simpler, we shall choose the phase plane analysis.

Since the remaining analysis and the results are exactly the same as those we followed and obtained respectively in the problem involving the discrete time version of the model in example 3 in Section 9.4.7, we do not repeat them here. Instead, we simply state here the steady-state values of $k, h,$ *and* y and present the associated phase plane. The steady-state values of $k, h,$ *and* y are $k(s) = [s_K^{1-\beta} s_H^{\beta}/(n+g)]^{1/(1-\alpha-\beta)}$,

$h(s)=[s_K^{\alpha}s_H^{1-\alpha}/(n+g)]^{1/(1-\alpha-\beta)}$, and $y(s)=[s_K^{1-\beta}s_H^{\beta}/(n+g)]^{\alpha/(1-\alpha-\beta)}[s_K^{\alpha}s_H^{1-\alpha}/(n+g)]^{\beta/(1-\alpha-\beta)}$, respectively. The phase-plane is illustrated in Figure 10.4.5.

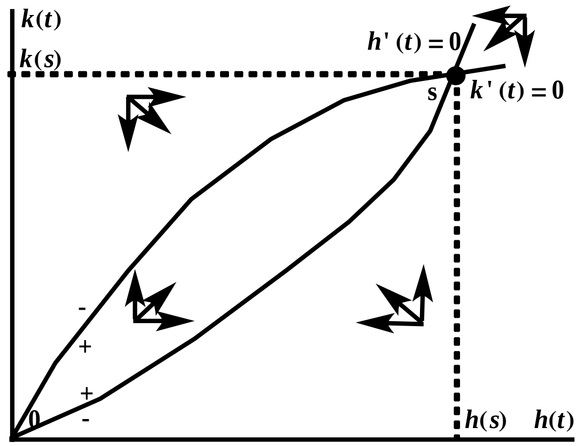


Figure 10.4.5

10.4.8 Exercises

1. Solve the following SSDLEs assuming that the constants are equal to one and determine their exact trajectories in the associated phase planes.

(i) $dy(t)/dt = 0\times y(t) - x(t) + 10$, $dx(t)/dt = -y(t) + 0\times x(t) + 10$ (ii) $dy(t)/dt = -y(t) + 0\times x(t) + 5$, $dx(t)/dt = y(t) - x(t) + 5$.

2. **Application exercise**. Suppose that the SSDLEs in example 1 in Section 10.4.7 takes the following form. Solve the system assuming that the constants are equal to one and determine the exact trajectory of the variables in a phase plane.

$dp(t)/dt = -2p(t) + F(t) + 15$ and $dF(t)/dt = p(t) - 2F(t) - 2.5$.

3. **Application exercise**. Suppose that the SSDLEs in the expected inflation and unemployment model we solved in example 2 in Section 10.4.7 takes the following form. Solve the system assuming that the constants are equal to one and determine the exact trajectory of the variables in a phase plane.

$p^e(t)/dt = -p^e(t) + 3u(t) - 5$ and $du(t)/dt = -p^e(t) - u(t) + 5$.

10.4.9 Mathematica applications

SSDLEs can be solved in Mathematica using the command **`DSolve`** introduced in Sections 10.2 and 10.3. The command **`DSolve[{`*eqn*$_1$`,`*eqn*$_2$`,...},{y`$_1$`[t], y`$_2$`[t],...},`*t*`]`** solves the SSDLEs **`{`*eqn*$_1$`,`*eqn*$_2$`,...}`** with independent variable **t**. As a demonstration, consider the SSDLEs we solved in equation (10.4.16): $dy(t)/dt = -2y(t) - 2x(t) + 80$ and $dx(t)/dt = 0\times y(t) - x(t) + 20$ with and without initial conditions [$y(0)=20$ *and* $x(0)=10$ when $t=0$]. This SSDLEs can be solved using the commands **`DSolve`** and **`Simplify`**, to obtain the same results as those obtained in Sections 10.4.2 and 10.4.3, as

```
Simplify[DSolve[{y'[t]==-2y[t]-2x[t]+80,x'[t]==0y[t]-
      x[t]+20},{y[t],x[t]},t]]
Simplify[DSolve[{y'[t]==-2y[t]-2x[t]+80,x'[t]==0y[t]-
      x[t]+20,y[0]==20,x[0]==10},{y[t],x[t]},t]]
```

$\{\{y[t]\to e^{-2t}(20e^{2t}+C[1]+2C[2]-2e^{t}C[2]),\ x[t]\to 20+e^{-t}C[2]\}\}$

$\{\{y[t]\to 20e^{-2t}(-1+e^{t}+e^{2t}),\ x[t]\to 20-10e^{-t}\}\}$.