CHAPTER 9

DIFFERENCE EQUATIONS

9.1 Introduction

In Chapter 8 we introduced the basics of integral calculus. We exposed the meaning, rules and properties of indefinite and definite integrals. We also exposed many applications of integral calculus in economics, business and finance. We found that many of the variables in these subjects were time-dependent. One of our major aims there was to determine the time paths of variables using integral calculus. But, as we stated at the end of Section 8.1, integral calculus is not the only mathematical method to determine the time paths of variables. There exist two other closely related methods for that purpose based on two branches of mathematics called difference equations and differential equations. We are now ready to turn our attention to the first of these methods.

Many of the relationships in the subjects of our interest are intertemporal in nature. The quantity demanded of various non-durable goods by a household during a particular time period depends, among others, on the quantity of the goods demanded and income saved in the previous time period(s). The quantity produced of many of the agricultural products during a particular time period depends on factors including the prices prevailed in the previous time period(s). The market value of a company during a particular time period depends to a great extent on factors including the market value of the company in the past time period. The productive capacity of an economy during a particular time period depends partly on the investments made in the economy in the previous time period(s). The quantity of a good produced by a firm during a time period depends, among others, on the size of the inventory in the previous time period. One can cite plenty of examples like these to illustrate the intertemporal nature of economic, business and financial relationships.

In integral calculus we found how we could delineate the time path of a variable when the variable is a function of time. But, the above description of the intertemporal nature of relationships in the fields of our interest shows that many of these relationships involves variables that are not the just the functions of time; they could be the functions of their own or other variables' past values too. An important question that arises at this juncture is how one can delineate the time paths of such variables. To answer this question we need to study either difference equations or differential equations.

The dependence of variables on their or other variables' past values through time may be either discrete or continuous. As an example, assume that the quantity produced of some agricultural cash crop in the current time period is a function of the crop's price in the last time period. It is sensible to consider the time period in this case as a year. If the last year's price of the crop was high enough to encourage the producers, then the producers are likely to produce a larger quantity of the crop this year. As another example, assume that the rate of inflation in an economy in the current time period may be dependent upon the change in the tax rate in the previous time period. In this example, it is sensible to consider the time period as a month. If the government changed the tax rate last month, then it is likely to take almost a month or two for the economy to experience the effect. As the last example, the price of a share or stock in a financial market during a particular time period may be dependent upon the price of the share in the market in the previous time period. In this third example, it is sensible to consider time period as a day or a week as share prices normally fluctuate daily or weekly.

In all the above examples the time that a variable takes to change were taken to be a year, or a month, or a week, or a day; and certainly not every possible point in time. In other words, in all these examples, time was considered to be discrete such that it takes zero time period, or one time period; or two time periods; and so on; that is, $t = 0, t = 1, t = 2,$ etc., where t denotes time. When we carry out an analysis, to determine the time paths of variables, in which we treat time as a discrete variable or as a series of distinct periods identifiable by integers, then that analysis is called a **discrete time analysis**. An important tool of discrete time analysis is a difference equation to which we turn our attention in the present chapter.

If, instead, the changes in variables were occurring at every possible point in time, then we would treat time as a continuous variable. When we carry out an analysis, to determine the time paths of variables, treating time as a continuous variable, then that analysis is called a **continuous time analysis**. An important tool of continuous time analysis is a differential equation. This is the topic of our discussion in Chapter 10.

9.2 First-Order Difference Equations

9.2.1 Definitions, notations and classification

We shall begin by defining a difference equation. A difference equation is defined as an equation that relates the value of a variable in a particular time period to its value in one or more other time periods and to the values of other variables. These values of the dependent variable form a sequence such as $y_t, y_{t+1}, y_{t+2},..., y_{t+i},..., y_{t+n}$ in time periods $t, t+1, t+2,..., t+i,..., t+n$, respectively. Suppose that the values that the variable y takes in two time periods t and $t+i$ are given by y_t and y_{t+i}. Then the difference between these two values of y during the two consecutive time periods is $y_{t+i} - y_t$ and is written as $\Delta y_t = y_{t+i} - y_t$. If we treat $i = 1$, then the difference in values of the dependent variable between y_t and $y_{t+i} = y_{t+1}$ can be written as $\Delta y_t = y_{t+1} - y_t$. Since the difference Δy_t is the difference between the values of the variable in two consecutive time periods, it is referred to as the **first difference**. Equations of the form $\Delta y_t = y_{t+1} - y_t$ are called difference equations.

Let us now present few examples of difference equations. Suppose that the difference between the quantities of output (y) produced of a cash crop in two consecutive time periods is given by

$$y_{t+1} = ay_t + b, \tag{9.2.1}$$

which is a difference equation. Notice that equation (9.2.1) can be written in an alternative but equivalent form as $y_t = ay_{t-1} + b$. Any equation that contains a difference term such as Δy is a difference equation. If the quantity of the crop produced in period $t+1$ also depends on time directly, then we write the difference equation as

$$y_{t+1} = ay_t + bt \quad . \tag{9.2.2}$$

As another example, suppose that the market value (v) of a company in time period $t+2$ is a function of the market values of the company in the previous two time periods. Then we can write the market value of the company in time period $t+2$ as

$$v_{t+2} = a_1 v_{t+1} + a_2 v_t + b, \tag{9.2.3}$$

which is a difference equation. If we assume that the market value of the company in period $t+2$ depends directly on time also, then we can write the difference equation as

$$v_{t+2} = a_1 v_{t+1} + a_2 v_t + bt \,. \tag{9.2.4}$$

As the third example, assume that the total capital (k) in an economy in time period $t+1$ is a function of the previous year's capital as given by

$$k_{t+1} = ak_t^{\alpha} + b, \tag{9.2.5}$$

which also represents a difference equation. If we find that the capital in the economy is also influenced by time directly, then we can write the difference equation as

$$k_{t+1} = ak_t^{\alpha} + bt \,. \tag{9.2.6}$$

As the last example, suppose that the portion of income saved (s) by a household in period $t+2$ is a function of the portions of savings by the household during the last two time periods and is given by

$$s_{t+2} = a_1 s_{t+1}^{\alpha} + a_2 s_t + b, \tag{9.2.7}$$

which is again a difference equation. If we assume that saving by the household is also influenced by time directly, then the difference equation can be written as

$$s_{t+2} = a_1 s_{t+1}^{\alpha} + a_2 s_t + bt \,. \tag{9.2.8}$$

Let us now present the **classification of difference equations** found in the literature. An important classification is based on the order with which difference equations appear. Consider first the difference equation (9.2.1): $y_{t+1} = ay_t + b$. Notice that the difference between the highest time period ($t+1$) and the lowest time period (t) in this equation is 1, or $t+1-t=1$. Therefore, this difference equation is called the **first-order difference equation**. Similarly, equations (9.2.2), (9.2.5) and (9.2.6) are also first-order difference equations. But the difference between the highest and the lowest time periods in equations (9.2.3), (9.2.4), (9.2.7) and (9.2.8) is 2 and, therefore, these equations are called the **second-order difference equations**. In general, if the difference between the highest time period and the lowest time period in a difference equation is n, then that difference equation is called a **n^th^-order difference equation**.

Another important classification of difference equations is based on whether they are linear or nonlinear. Consider first the difference equation (9.2.1). In this equation the y term, irrespective of the time period it represents, appears with the power one and is not multiplied or divided by other y terms that represent other time period(s). Therefore, this difference equation is called a **linear difference equation**. In general, if a difference equation contains variables that are raised to power 1 and are not multiplied or divided by the same or any other variable that represents the same or other time period(s), then it is called a linear difference equation. Otherwise, the difference equation is called a **nonlinear difference equation**. If we apply this definition, we see that equations (9.2.2) through (9.2.4) are linear difference equations; and equations (9.2.5) through (9.2.8) are called nonlinear difference equations if $\alpha \neq 1$.

The last classification of difference equations we make here is based on the nature of the coefficients and terms in the equations. If the coefficient a and the term b in equations (9.2.1) through (9.2.8) are constants, then these equations are called **autonomous difference equations**. If in these equations $b=0$, then we call these equations **homogenous difference equations**; and if $b \neq 0$, then they are called **non-homogenous difference equations**. If, in any difference equation, either the coefficient a, or the term b, or both, are some functions of time, then that equation is called a **non-autonomous difference equation**.

9.2.2 First-order, linear, autonomous difference equations: the iterative solution

Suppose that we have a difference equation of the form in equation (9.2.1): $y_{t+1} = ay_t + b$. Notice that this is a linear, first-order, autonomous difference equation. Our aim here is to find out the time path of the variable of the equation (y_t); that is, to solve the difference equation. Solving a difference equation means determining the underlying function of time that generated the equation. There are two popular methods to solve a difference equation: the **iterative method** and the **matrix method**. We will expose the iterative method in the present section and the matrix method will be exposed in the following section.

Let us begin as follows. Given the difference equation $y_{t+1} = ay_t + b$, we have $y_1 = ay_0 + b$ in the initial time period or when $t=0$. When $t=1$, the difference equation can be written as $y_2 = ay_1 + b$. Substituting the result $y_1 = ay_0 + b$ into $y_2 = ay_1 + b$ we obtain $y_2 = ay_1 + b = a[ay_0 + b] + b = a^2 y_0 + ab + b$. When $t=2$, the difference equation can be written as $y_3 = ay_2 + b$. Substituting $y_2 = a^2 y_0 + ab + b$ into $y_3 = ay_2 + b$ we obtain $y_3 = ay_2 + b = a[a^2 y_0 + ab + b] + b \quad = a^3 y_0 + a^2 b + ab + b = a^3 y_0 + b[a^2 + a + 1]$. If we continue this iterative procedure for t time periods, we can write y_t as

$$y_t = a^t y_0 + b[a^{t-1} + a^{t-2} + ... + a^2 + a + 1] = a^t y_0 + b[1 + a + a^2 + ... + a^{t-2} + a^{t-1}] \ . \qquad (9.2.9)$$

Notice that the terms inside the brackets in equation (9.2.9) is a geometric series with a as the common ratio and 1 as the first term. Therefore, we can apply equation (1.10.7) to obtain $1 + a + a^2 + ... + a^{t-2} + a^{t-1} = (1-a^t)/(1-a)$. Substituting this result into equation (9.2.9) we obtain the solution to the difference equation (9.2.1) as

$$y_t = a^t y_0 + b[1 + a + a^2 + ... + a^{t-2} + a^{t-1}] = a^t y_0 + b\left[\frac{1-a^t}{1-a}\right]. \quad (9.2.10)$$

Notice that this solution satisfies both the difference equation and the initial condition $y_t = y_0$ when $t = 0$. The solution in equation (9.2.10) can be also be written as

$$y_t = a^t y_0 + b\left[\frac{1-a^t}{1-a}\right] = \left[y_0 - \frac{b}{1-a}\right]a^t + \frac{b}{1-a} \qquad \text{if } a \neq 1\text{, and} \quad (9.2.11)$$

$$y_t = y_0 + bt \quad \text{if } a = 1. \quad (9.2.12)$$

Notice that if $b = 0$ (that is, when the difference equation is autonomous and homogenous) we have from equations (9.2.11) and (9.2.12) that $y_t = a^t y_0$ if $a \neq 1$ and $y_t = y_0$ if $a = 1$, respectively. One question arising now is what happens to y_t in equation (9.2.10) when $t \to \infty$ and when $|a| < 1$. Let us denote the value of y_t in equation (9.2.10) when $t \to \infty$ and when $|a| < 1$ by $\lim_{t\to\infty} y_t = y_s$. We can find this value by applying equation (1.10.10). Therefore, we obtain the result

$$\lim_{t\to\infty} y_t = y_s = b/(1-a). \quad (9.2.13)$$

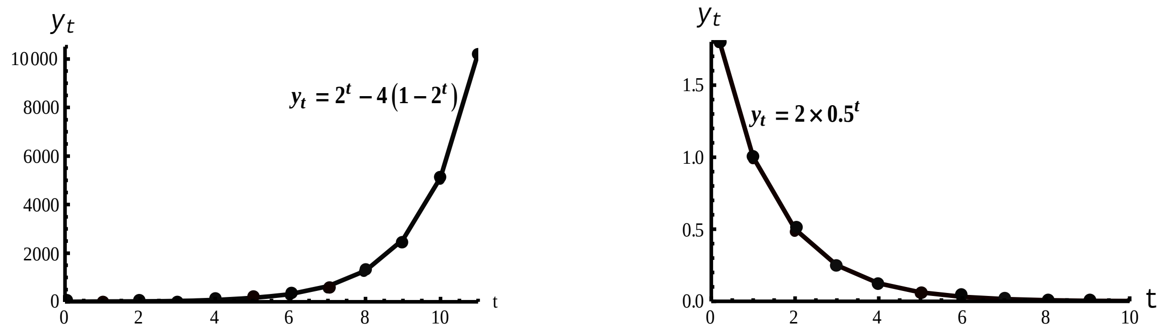
This result implies that, when $t \to \infty$ and when $|a| < 1$, y_t converges to a constant value $y_s = b/(1-a)$. One may also ask another question: What happens to y_t when $t \to \infty$ and when $|a| > 1$. We shall denote the value of y_t in equation (9.2.10) when $t \to \infty$ and when $|a| > 1$ by $\lim_{t\to\infty} y_t = y_d$. Therefore, when $t \to \infty$ and when $|a| > 1$, we have the result

$$\lim_{t\to\infty} y_t = y_d = \lim_{t\to\infty}\left(a^t y_0 + b\left[\frac{1-a^t}{1-a}\right]\right) = \frac{b}{1-a} + \left[y_0 - \frac{b}{1-a}\right]\lim_{t\to\infty} a^t, \quad (9.2.14)$$

which diverges when $t \to \infty$.

As an example, consider the equation $y_{t+1} = 2y_t + 4$ with the initial condition that $y_0 = 1$. Notice that this is a linear, first-order, autonomous, non-homogenous difference equation. Notice also that this equation is identical with equation (9.2.1) if we treat $a = 2$ and $b = 4$. Then, applying equation (9.2.10), we obtain the solution as $y_t = y_0 2^t + 4[(1-2^t)/(1-2)] = y_0 2^t - 4(1-2^t) = 1 \times 2^t - 4(1-2^t) = 2^t - 4(1-2^t)$. We can verify this result by substituting $t = 0$ and $t = 1$ into the solution $y_t = 2^t - 4(1-2^t)$. With $t = 0$ and $t = 1$, we obtain $y_0 = 1$ and $y_1 = 6$, respectively. Notice that the original equation $y_{t+1} = 2y_t + 4$ can be written in the alternative but equivalent form $y_t = 2y_{t-1} + 4$. Let us now substitute the two values $y_0 = 1$ and $y_1 = 6$ into the last, alternative form. If we substitute $y_0 = 1$ for y_{t-1} and $y_1 = 6$ for y_t, we find that $6 = 2 \times 1 + 4 = 6$ and, therefore, the solution is correct. This confirms the fact that the solution to a difference equation must satisfy the equation for all values of t.

Let us now plot the solution $y_t = 2^t - 4(1-2^t)$ to the difference equation $y_{t+1} = 2y_t + 4$ as illustrated in Panel (A) of Figure 9.2.1. We stated above that when $t \to \infty$ and when $|a| > 1$ [as can seen from equation (9.2.14)] y_t would diverge. This is confirmed by the graph of y_t for $|a| = |2| = 2 > 1$ in this panel.



(A) (B)

Figure 9.2.1

As another example, consider the equation $y_{t+1} = 0.5y_t$ with the initial condition $y_0 = 2$. Notice that this is a linear, first-order, autonomous, homogenous difference equation. Notice also that this equation is identical with equation (9.2.1) if we treat $a = 0.5$ and $b = 0$. Then, applying equation (9.2.10), we obtain the solution $y_t = y_0(0.5)^t + 0\times[(1-0.5^t)/(1-0.5)] =$ $y_0(0.5)^t = 2\times(0.5)^t$. As before, we can verify this result by substituting $t = 0$ and $t = 1$ into the solution. With $t = 0$ and $t = 1$, we obtain $y_0 = 2$ and $y_1 = 1$, respectively. Notice that the original difference equation $y_{t+1} = 0.5y_t$ can be written in the alternative but equivalent form $y_t = 0.5y_{t-1}$. Let us now substitute the two values $y_0 = 2$ and $y_1 = 1$ into the last difference equation. If we substitute $y_0 = 2$ for y_{t-1} and $y_1 = 1$ for y_t, we find that $1 = 2\times 0.5 = 1$ and, therefore, the solution is correct. This again confirms the fact that the solution to a difference equation must satisfy the equation for all values of t.

As in the case of the last example, let us plot the above solution as illustrated in Panel (B) of Figure 9.2.1. We mentioned earlier that when $t \to \infty$ and when $|a|<1$, y_t would converge. This is confirmed by the graph of y_t for $|a| = |0.5| = 0.5 < 1$ in this panel. In the present example y_t converges to zero.

9.2.3 First-order, linear, autonomous difference equations: the general solution

In the last section we determined, as given in equation (9.2.10), the solution to the linear, first-order, autonomous (homogenous or non-homogenous) difference equation (9.2.1). We also found in the last section that this was the only one solution that satisfied both the difference equation and the initial condition together.

Suppose that we have a variant of the difference equation (9.2.1) with $b = 0$. Then equation (9.2.1) becomes

$$y_{t+1} = ay_t. \qquad (9.2.15)$$

Notice that equation (9.2.15) is a first-order, linear, autonomous, homogenous difference equation. Our aim is to find the solution to this equation. We know, from the last section, that a solution to this equation is a function of t that satisfies the equation for any value of t. We can find the solution to equation (9.2.15) by applying equation (9.2.10) as: $y_t = a^t y_0 + b[1 + a + a^2 + ... + a^{t-2} + a^{t-1}] = a^t y_0 + b[(1-a^t)/(1-a)]$ $= a^t y_0 + 0\times[(1-a^t/(1-a)]$ $= a^t y_0$. Notice that we did not specify any initial condition in this solution. This solution is called the **complementary function**. Treating $C = y_0$, the last result can be written as $y_c = Ca^t$, where the subscript c denotes that the result represents the complementary function of the solution.

Suppose now that we have a first-order, linear, autonomous, non-homogenous difference equation of the form

$$y_{t+1} = ay_t + b, \qquad (9.2.16)$$

which is identical with equation (9.2.1). How do we solve this general difference equation? The solution to this general difference equation, called the **general solution** or the **complete solution**, comprises of two parts. One part is the complementary function referred to above. The other part is called the **particular solution**. The complementary function is already obtained as $y_c = Ca^t$. Let us now find the particular solution. The particular solution is any solution that satisfies the difference equation for every t. We shall try the simplest possible particular solution $y_t = p$, where p is a constant. Since p is constant, we have $y_{t+1} = p$. Let us now substitute these values in equation (9.2.16) to obtain $p = ap + b$. Solving the last equation for p yields $p = b/(1-a)$. Since this result satisfies equation (9.2.16), the particular solution can be written as

$$y_s = p = b/(1-a), \tag{9.2.17}$$

where the subscript 's' denotes particular solution. Notice that the condition for the existence of the result in equation (9.2.17) is $a \neq 1$. If $a = 1$, we will have to try forms, instead of $y_t = p$, such as $y_t = pt$, $y_t = pt^2$, etc. Therefore, we can now write the complete solution to equation (9.2.16) as the sum of the complementary function and the particular solution as $y_t = y_c + y_s$, which can be written as

$$y_t = y_c + y_s = Ca^t + \frac{b}{1-a} \quad \text{if} \quad a \neq 1, \text{ and} \tag{9.2.18}$$

$$y_t = y_c + y_s = C + bt \quad \text{if} \quad a = 1. \tag{9.2.19}$$

Notice that it is difficult to determine the above two equations completely due to the presence of the constant C. We can use the initial condition $y_t = y_0$ when $t = 0$ to obtain the **definite solution**. Setting $t = 0$ in the last two equations we obtain $y_t = [C - \{b/(1-a)\}]a^t + b/(1-a) = C - [b/(1-a)] + [b/(1-a)] = y_0$, or $y_t = C = y_0$. With this initial condition, we can convert the above general and complete solutions to definite and complete solutions as $y_t = y_0 a^t + b[(1-a^t/(1-a)] = [y_0 - \{b/(1-a)\}]a^t + b/(1-b)$ if $a \neq 1$ and $y_t = y_0 + bt$ if $a = 1$, which are identical with definite solutions in equations (9.2.11) and (9.2.12), respectively.

As an example, consider the equation $y_{t+1} = 5y_t + 6$. Notice that this is a linear, first-order, autonomous, non-homogenous difference equation and is identical with equation (9.2.1) with $a = 5$ and $b = 6$. Notice also that we did not specify any initial condition. This implies that we can apply the result in equation (9.2.18) to find the general solution to the equation. The complementary function (y_c) is the solution to the homogenous form $y_{t+1} = 5y_t$ of the given equation. Therefore, in the present example we obtain $y_c = Ca^t = C5^t$; and the particular solution, applying equation (9.2.17), can be written as $y_s = b/(1-a) = 6/(1-5) = 6/(1-5) = -1.5$. Thus, applying equation (9.2.18), we obtain the complete solution to the equation as $y_t = y_c + y_s = Ca^t + b/(1-a) = C5^t - 1.5$.

As another example, consider the equation $y_{t+1} = y_t + 6$. Notice that this is a linear, first-order, autonomous, non-homogenous difference equation and is identical with equation (9.2.1) if we treat $a = 1$ and $b = 6$. As before, we did not specify any initial condition in this problem. This implies that we can apply the result in equation (9.2.19) to find the general solution to the equation. The complementary function in the present example is $y_c = Ca^t = C \times 1^t = C$. To find the particular solution, let us set $y_t = pt$ (instead of $y_t = p$, as we did above) because $a = 1$. Substituting $y_t = pt$ into the equation, we obtain $p(t+1) = pt + 6$, which when solved for p (with $a = 1$) will yield $p = 6$. This gives us the particular solution $y_s = 6$. We can now apply equation (9.2.19) to obtain the complete solution as $y_t = y_c + y_s = C + bt = C + 6t$.

Let us now convert equation (9.2.11) or equation (9.2.18), for analytical convenience, into the form

$$y_t = Aa^t + d\ , \qquad (9.2.20)$$

where $A = C - b/(1-a)$ without initial conditions and $A = y_0 - b/(1-a)$ with initial condition, and $d = b/(1-a)$. Notice that when the difference equation is homogenous (that is, when $b = 0$) equation (9.2.20) or (9.2.18) reduces to $y_t = Aa^t = Ca^t$. Notice also that in equation (9.2.20) the first term on the R.H.S. (Aa^t) is the complementary function and the last term (d) is the particular solution. Therefore, equation (9.2.20) divides the time path of y into two parts: one is the complementary function and the other is the particular solution. As we shall see below, the latter shows the intertemporal or long-run equilibrium value of y while the former represents the deviations from that value.

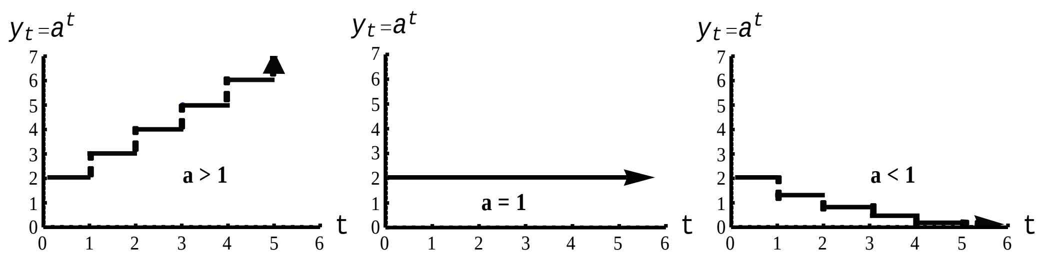
9.2.4 Stability, steady-state and convergence

We found in the last section that, given the difference equation $y_{t+1} = ay_t + b$, the complete solution was given by equation (9.2.20): $y_t = Aa^t + d$. One important question at this point is what happens to y_t as $t \to \infty$ or what happens to the path of the dependent variable (y) in the long-run. Will y_t remain stable (converge to an equilibrium or a constant value) or unstable (diverge) in the long-run? What conditions do cause the time path to be stable or unstable? Let us attempt to answer these questions in the present section. The value at which the path of y remains stable is called the **steady-state value** or the **stationary value** or **long-run equilibrium value**.

We have already mentioned the concepts of convergence and divergence when we plotted the time paths of the solutions $y_t = 2^t - 4(1-2^t)$ and $y_t = 2 \times (0.5)^t$ to the difference equations $y_{t+1} = 2y_t + 4$ and $y_{t+1} = 0.5y_t$, respectively, in Panels (A) and (B), respectively, of Figure 9.2.1. We found that $y_t = 2^t - 4(1-2^t)$ was diverging and that $y_t = 2 \times (0.5)^t$ was converging. The reasons we stated for divergence and convergence were that the absolute values of the coefficients in the difference equations were greater than one and less than one (that is; $|a| > 1$ and $|a| < 1$, along with $t \to \infty$), respectively. Notice that this conclusion was based on specific examples of difference equations and their solutions mentioned above.

Let us now consider a general difference equation such as the one in equation (9.2.1): $y_{t+1} = ay_t + b$, the general solution to which is presented in equation (9.2.18): $y_t = Ca^t + b[(1-a^t)/(1-a)]$. The converted form of this solution was presented in equation (9.2.20): $y_t = Aa^t + d$. Notice that in the last equation the time path y_t depends on two factors: the complementary function Aa^t and the particular solution d . Assume, for convenience, that $A = 1$ and $d = 0$. Then the time path can be written as $y_t = a^t$, which implies that the long-run behavior of y_t depends only on the value a .

Notice that a can assume values such that $-\infty \le a \le +\infty$ or, more specifically, $a > 1$, $a = 1$, $0 < a < 1$, $a = 0$, $-1 < a < 0$, $a = -1$, and $a < -1$. Depending upon which of these seven values that a may take we will obtain seven different time paths for the dependent variable y_t . These time paths are illustrated in Panels (A) through (G) of Figure 9.2.2.



(A) (B) (C)

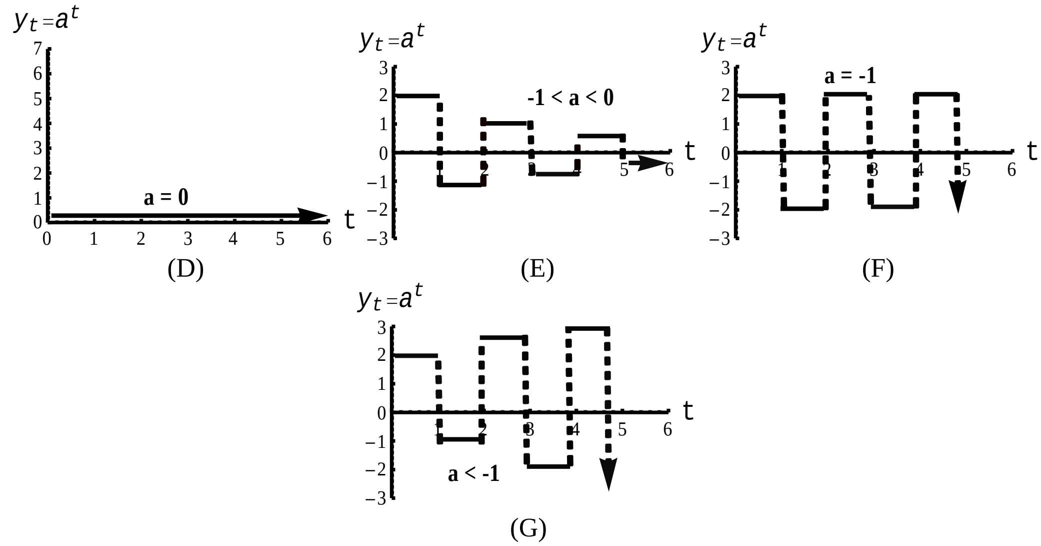


Figure 9.2.2

As can be seen from Panel (A) of Figure 9.2.2, if $a > 1$ the time path of the dependent variable y_t diverges; moving farther and farther from the horizontal axis as time passes. If $a = 1$, then the time path moves on a straight line parallel to the horizontal axis as is illustrated in Panel (B). Panel (C) shows the graph of the path when a lies in between zero and one. When $0 < a < 1$, the time path tends to converge, moving closer and closer to the horizontal axis. If $a = 0$, the time path coincides with the horizontal axis, as is illustrated in Panel (D). If $-1 < a < 0$, the time path will take alternative signs but will move closer and closer to the horizontal axis or tends to converge to 0 as is illustrated in Panel (E). Panel (F) shows the case when $a = -1$. When $a = -1$, the time path oscillates between +1 and – 1 and, therefore, there is neither convergence nor divergence. If $a < -1$, as shown in Panel (G), the time path oscillates between positive and negative values but moves farther and farther from the horizontal axis. In other words, if $a < -1$, the time path diverges from the horizontal axis.

Let us now summarize the above results. If the absolute value of the coefficient a is greater than 1 (that is, if $|a| > 1$), then the time path will diverge from the horizontal axis as shown in Panels (A) and (G) in Figure 9.2.2. If the absolute value of a is less than 1 (that is, $|a| < 1$), the time path will converge to the horizontal axis as shown in Panels (C) and (E). Notice that these conclusions are identical with the conclusions we obtained from graphs in Figure 9.2.1. If $a = 1$ or if $a = 0$, the time path remains constant as shown in Panels (B) and (D). If $a = -1$, the time path oscillates between identical negative and positive values as is depicted in Panel (F). Finally, if $a > 0$ the time path does not oscillate [Panels (A) through (C)]; and if $a < 0$, the time path oscillates between negative and positive values [Panels (E) through (G)].

As an example, consider the equation $y_{t+1} = -0.75 y_t$. Notice that this is a linear, first-order, autonomous, homogenous difference equation and is identical with equation (9.2.1) with $a = -0.75$ and $b = 0$. Therefore, we may apply equation (9.2.20) with $d = 0$ to find the solution to this equation. Applying equation (9.2.20) we obtain the solution as $y_t = Aa^t$, where $A = C$. Assuming $A = 1$, the solution can be written as $y_t = a^t$. Since $a = -0.75$, the solution can be explicitly written as $y_t = a^t = (-0.75)^t$. Since $-1 < a = -0.75 < 0$, $y_t = a^t = (-0.75)^t$ will take alternative signs (as t increases) but will move closer and closer to zero or tends to converge to the horizontal axis. This time path is illustrated in Panel (A) of Figure 9.2.3. Notice that the graph of the time path in this panel is similar to that in Panel (E) in Figure 9.2.2.

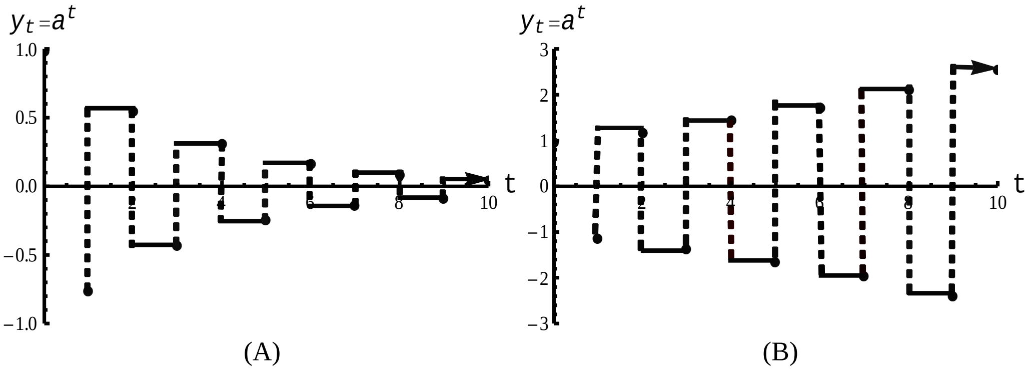


Figure 9.2.3

As another example, consider the equation $y_{t+1} = -1.1y_t$. Notice that this is a linear, first-order, autonomous, homogenous difference equation and is identical with equation (9.2.1) with $a = -1.1$ and $b = 0$. Therefore, applying equation (9.2.20) with $d = 0$ we obtain the solution $y_t = Aa^t$, where $A = C$. Assuming $A = 1$, the solution can be written as $y_t = a^t$. Since $a = -1.1$, the solution can be explicitly written as $y_t = a^t = (-1.1)^t$. Since $a = -1.1 < -1$, $y_t = (-1.1)^t$ will take alternative signs as t increases but will move farther and farther, or tends to diverge, from the horizontal axis. This time path is illustrated in Panel (B) of Figure 9.2.3. Notice that the graph of the time path in this panel is similar to that in Panel (G) of Figure 9.2.2.

So far we were attempting to determine the path of the solution $y_t = Aa^t + d$ in equation (9.2.20) to the difference equation (9.2.1) assuming that $A = 1$ and $b = 0$. Let us now treat $A \neq 1$ and continue to assume $b = 0$ (and, therefore, $d = 0$). What will be the effect on the path of the variable y_t of treating $A \neq 1$? There will be two effects of treating $A \neq 1$. One will be due to the size of A and the other will be due to the sign of A. The former is expected to generate a **scale effect** pushing the path up or down, and the latter is expected to generate both a scale effect and a **mirror effect**, giving a path that is a mirror image of the former.

As an example, consider again the solution $y_t = Aa^t = A(-0.75)^t$ above to the difference equation $y_{t+1} = -0.75y_t$. Notice that we treated $A = 1$ in this solution. Let us now treat $A = 5$ and $A = -5$. The impacts of treating $A = 1$, $A = 5$ and $A = -5$ on the path of $y_t = Aa^t = A(-0.75)^t$ are illustrated in Panel (A) of Figure 9.2.4. The graphs in this panel are plotted, for convenience, by joining the values of the dependent variable y_t when the independent variable t takes discrete values from 0 to 10.

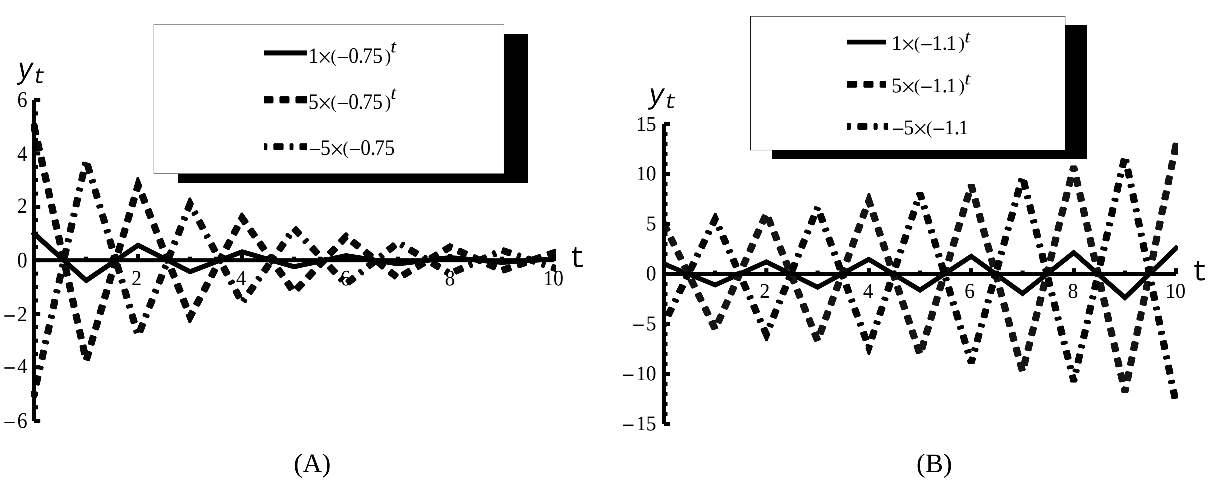


Figure 9.2.4

As can be seen from Panel (A) of Figure 9.2.4, the solid graph represents the path of $y_t = Aa^t = 1\times(-0.75)^t$; that is the path when $A = 1$. This path is identical with the path illustrated in Panel (A) of Figure 9.2.3. But, the path of $y_t = 5(-0.75)^t$, when $A = 5$, is 5 times the path of $y_t = 1 \times (-0.75)^t$, and is illustrated in the panel with dashed line. This path pulls up the path of $y_t = 1 \times (-0.75)^t$ by 5 times. This was what we referred to above as the scale effect. Similarly, the path of $y_t = -5(-0.75)^t$, when $A = -5$, pulls down the path of $y_t = 1 \times (-0.75)^t$ by -5 times, and is illustrated in the panel with dashed-dotted line. But, this is the opposite of the path of $y_t = 5(-0.75)^t$. Although this too is a scale effect, it is in the opposite direction generating a mirror image of the path of $y_t = 5(-0.75)^t$. This was the reason why we referred to it above as the mirror effect.

As another example, consider the solution $y_t = a^t = (-1.1)^t$ to equation $y_{t+1} = -1.1 y_t$ obtained earlier. Notice that we treated $A = 1$ in this solution. Let us now treat $A = 5$ and $A = -5$. The impacts of treating $A = 1$, $A = 5$ and $A = -5$ on the path of $y_t = a^t = (-1.1)^t$ are illustrated in Panel (B) of Figure 9.2.4. The explanations of these time paths are similar to those given above in the case of the time paths illustrated in Panel (A) of the same figure.

So far we were attempting to see the impacts when the terms A and a in equation (9.2.20) assume different values. In other words, so far we were concerned with the impacts of changes in the parameters of the complementary function (Aa^t) on the path of $y_t = Aa^t + d$ and left out, for convenience, the particular solution d. Suppose that, in equation (9.2.1) or (9.2.18), $b \neq 0$, which means that, in equation (9.2.20), $d \neq 0$. Therefore, we have to take into account the effect of adding the term d on the time path. What will be the effect of adding the particular solution to the path of the dependent variable of the solution? The answer to this question is as follows.

When we assumed $b = 0$, and thereby $d = 0$, our reference point of convergence (or steady-state) or divergence was the horizontal axis. That is, our concern was whether the time path converged to or diverged from the x-axis. Contrary to this, our reference point of convergence or divergence will be the particular solution (the constant value d) when we add this term to the time path. In other words, the addition of the particular solution will in no way alter the time path, but will only alter the reference point of the steady-state.

9.2.5 The phase diagrams

So far in the present chapter we were attempting to solve the first-order, linear, autonomous, homogenous or non-homogenous difference equations explicitly. We were also interested in the behavior of the time paths of these equations mainly to see whether they converge to, or diverge from, or oscillate around, the long-run equilibrium or steady-state or stationary value (horizontal axis when $b = d = 0$ and d when $b \neq 0$). For this we plotted the graphs of the solutions to a number of difference equations.

There exists another geometric way to observe the long-run behavior of the solution to a difference equation. This is based on a diagram called **phase diagram**. This method is particularly useful when the difference equation is nonlinear or when there is a **simultaneous system of difference equations**, a topic we will discuss later. Suppose that we have a difference equation of the form of (9.2.1): $y_{t+1} = ay_t + b$. In a phase diagram we consider y_{t+1} as a function of y_t; $y_{t+1} = f(y_t)$. We first draw the graph of the equation and it is called the **phase line**. Then we draw a 45° line. We can then pick up an initial value of y_t, denoted by y_0, and then map y_t into y_{t+1} using the function and the 45° line. Lastly, we connect the phase line and the 45° line generated by the mapping process by arrows, which can be used to judge the convergent or divergent behavior of the underlying variable. The point at which the phase line crosses the 45° line will be an equilibrium or steady-state point (y_s). If a difference equation has a solution which will converge to (diverge from) the steady-state, then the convergence (divergence) will occur irrespective of the fact that the initial value we choose is above or below the equilibrium value (that is, $y_0 < y_s$ or $y_0 > y_s$).

Let us use four imaginary difference equations represented by the four functions $y_{t+1} = f^1(y_t)$, $y_{t+1} = f^2(y_t)$, $y_{t+1} = f^3(y_t)$, and $y_{t+1} = f^4(y_t)$. We can now map y_t using the function $f^{i}(y_t)$, where $i = 1,2,3,$ *and* 4, to obtain the phase lines illustrated in Panels (A) through (D) of Figure 9.2.5.

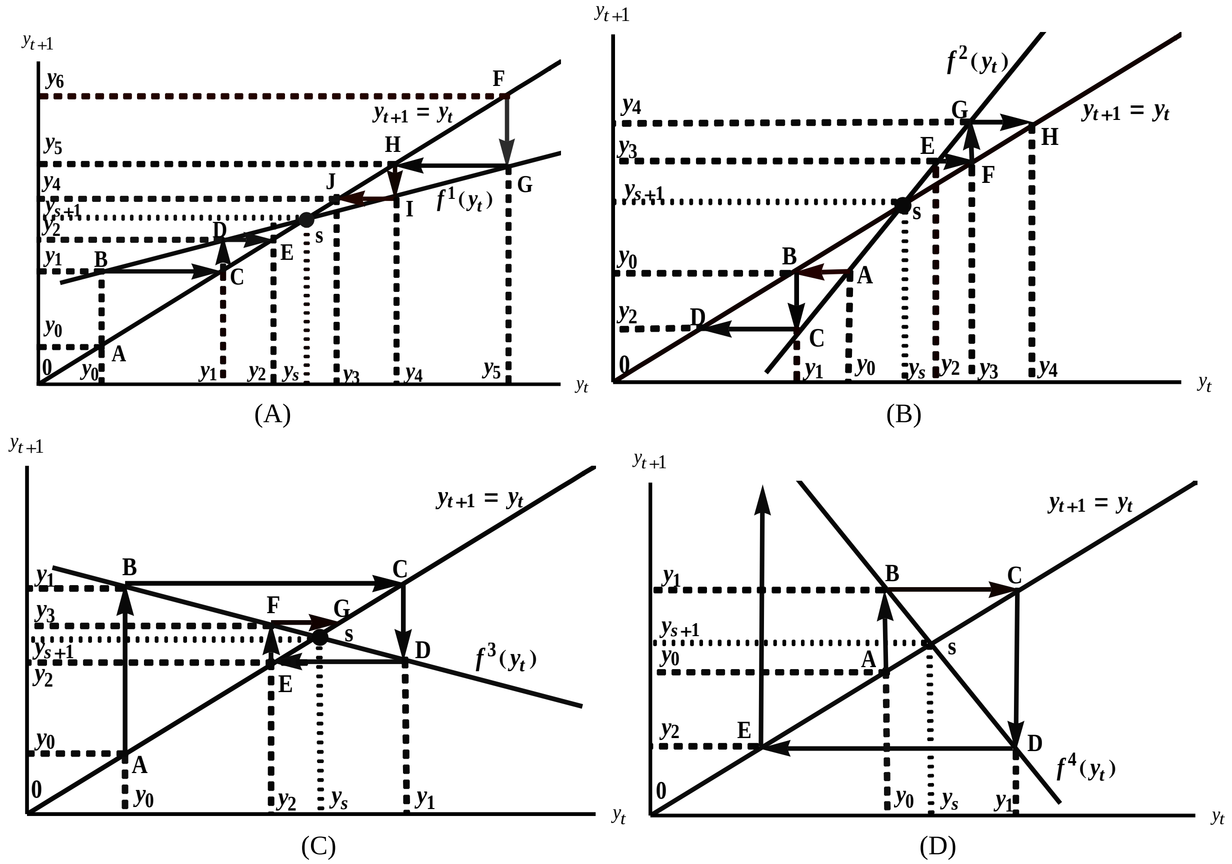


Figure 9.2.5

Let us now explain the phase diagrams in Panels (A) through (D) of Figure 9.2.5 beginning with the one in Panel (A). We first choose an initial value of y_t, say $y_0 < y_s$. Let us now map this value y_0, using the function $y_{t+1} = f^{1}(y_t)$ to obtain the point y_1 on the vertical axis, which gives us the coordinate point B or (y_0, y_1). Then we can pick up another point, say y_1 on the horizontal axis and map it to obtain the point y_2 on the vertical axis, which gives us the coordinate point D or (y_1, y_2). Once we continue like this, we will obtain a number of coordinate points such as A, B, C, D, or E. Connecting all these coordinate points with arrows, we obtain the arrows illustrated in the panel. These arrows show the path of the variable y_t and that path approaches the steady-state or equilibrium point s as mapping of y_t into y_{t+1} approaches y_s. Notice that we would obtain a similar result had we chosen an initial value $y_0 > y_s$. Therefore, the time path converges to the steady-state value y_s irrespective of the initial value chosen, and at the steady state (the point where the phase line crosses the 45° line) is $y_{t+1} = y_t = y_s$.

One can carry out a similar procedure to obtain the phase diagrams illustrated in Panels (B) through (D) of Figure 9.2.5. As can be seen from this figure, the time paths diverges, converges (with oscillation) and diverges (with oscillation) as illustrated in Panels (B), (C), and (D), respectively. Again, the time path converges to or diverges from the steady-state value irrespective of the initial value chosen, and at the steady state we have $y_{t+1} = y_t = y_s$.

An important question now is why the time paths illustrated in Panels (A), (B), (C), and (D) of Figure 9.2.5 converged to, diverged from, converged with oscillation to, and diverged with oscillation from their respective steady-states. The answer depends on the signs and absolute values of the slopes of the phase lines illustrated in the figure. The slope of the phase line in Panel (A) is positive but lies in between zero and one; that is, $0 < f^{1\prime}(y_t) < 1$. If the slope of the phase line lies in between zero and one, then the time path of the variable will converge to the steady-state. The slope of the phase line in Panel (B) is greater than one; that is, $f^{2}{}'(y_t) > 1$; and, therefore, the time path of the variable diverges from the steady-state. But, the slope of the phase line in Panel (C) lies in between -1 and 0; that is, $-1 < f^{3}{}'(y_t) < 0$. If the slope of the phase line lies in between -1 and zero, then the time path of the variable will converge to the steady-state with oscillation. Finally, the slope of the phase line in Panel (D) is less than -1; that is, $f^{4}{}'(y_t) < -1$ and, therefore, the time path of the variable diverges from the steady-state with oscillation. In short, if the slope of a phase line lies in between 0 and +1 (-1 and 0), then the time path represented by the phase line will converge (converge with oscillation) to the steady-state; and if the slope of the phase line is greater than 1 (less than -1), then the time path will diverge (diverge with oscillation) from the steady-state. Therefore, the absolute value of the slope of the phase line determines convergence or divergence and its sign determines oscillation or non-oscillation. Notice that this conclusion is in accordance with our conclusion based on the time paths illustrated in Panels (A), (C), (E), and (G) of Figure 9.2.2.

As an example, consider the difference equation $y_{t+1} = -0.75y_t + 7$. In terms of the result we obtained above, the path of the underlying variable (y_t) must converge to steady-state with oscillation because the first derivative of y_{t+1} with respect to y_t is negative and its absolute value is in between 0 and 1; that is, $d(y_{t+1})/dy_t = f'(y_t) = -0.75 < 0$ and 0 < 0.75< 1, respectively. How do we show that the underlying variable y_t of this difference equation converges (with oscillation) without solving it explicitly? Here lies the use of phase diagram. Notice that the difference equation $y_{t+1} = -0.75y_t + 7$ is of the functional form $y_{t+1} = f(y_t) = 7 - 0.75y_t$. Notice also that $y_{t+1} = y_t = y_s$ in the steady-state. Therefore, substituting $y_{t+1} = y_t$ into $y_{t+1} = f(y_t) = 7 - 0.75y_t$ we can solve for the steady-state value of y. This steady-state value is $y_{s+1} = y_s = 4$. Let us now follow the same procedure as the one followed in the construction of the phase diagrams in Figure 9.2.5 to construct the phase diagram for the difference equation $y_{t+1} = f(y_t) = 7 - 0.75y_t$ as illustrated in Panel (A) of Figure 9.2.6.

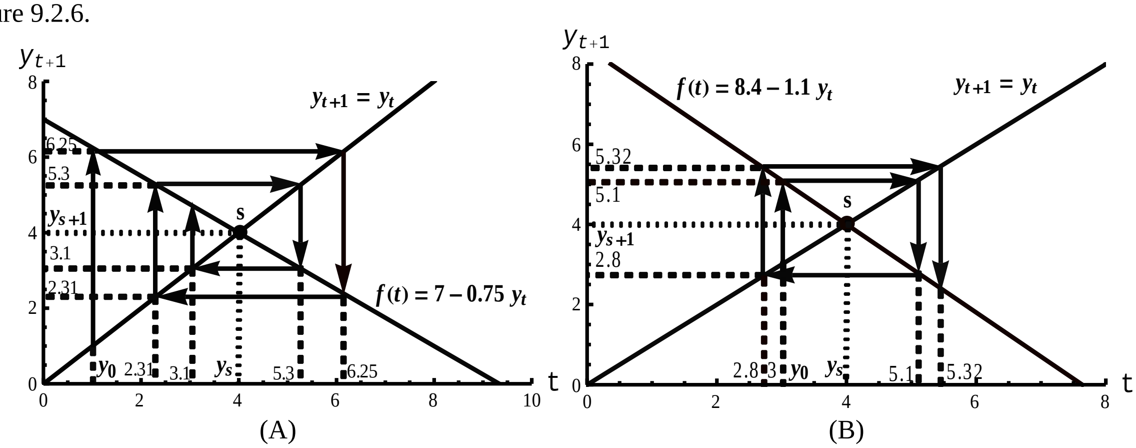


Figure 9.2.6

Suppose that we choose the initial value of y_t as $y_0 = 1$ in the phase diagram of the difference equation $y_{t+1} = 7 - 0.75y_t$ illustrated in Panel (A) of Figure 9.2.6. When this value is mapped into y_{t+1} using the equation, we obtain $y_1 = 6.25$ on the vertical axis. But this value of y_{t+1} corresponds to $y_2 = 6.25$ of y_t. For $y_2 = 6.25$ of y_t, the value of y_{t+1} is $y_3 = 2.31$. If we continue like this, we obtain the complete oscillation and convergence (to the steady-state value 4) depicted by the movements of the arrows in this panel. Notice that we would have obtained a similar result had we chosen an initial value above $y_s = 4$ (that is, $y_0 > y_s = 4$).

As another example, consider the difference equation $y_{t+1} = -1.1y_t + 8.4$. In terms of the result we obtained earlier, the path of the underlying variable (y_t) must diverge from the steady-state with oscillation because the first derivative of y_{t+1} with respect to y_t is negative and its absolute value is larger than one; that is, $d(y_{t+1})/dy_t = f'(y_t) = -1.1 < 0$ and 1.1 > 1, respectively. Our task here is to show that the underlying variable y_t of this difference equation diverges (with oscillation) without solving it explicitly. Notice that the difference equation $y_{t+1} = -1.1y_t + 8.4$ is of the functional form $y_{t+1} = f(y_t) = 8.4 - 1.1y_t$. Notice also that $y_{t+1} = y_t = y_s$ in the steady-state. Therefore, substituting $y_{t+1} = y_t$ into $y_{t+1} = g(y_t) = 8.4 - 1.1y_t$ we can solve for the steady-state value of y, which is $y_{s+1} = y_s = 4$. Let us now follow the same procedure exposed above and construct the phase diagram for the difference equation $y_{t+1} = f(y_t) = 8.4 - 1.1y_t$ as illustrated in Panel (B) of Figure 9.2.6.

Suppose that we choose, in the phase diagram illustrated in Panel (B) of Figure 9.2.6, the initial value of y_t as $y_0 = 3$. When this value is mapped into y_{t+1} using $y_{t+1} = 8.4 - 1.1y_t$, we obtain $y_1 = 5.1$ on the vertical axis. But this value of y_{t+1} corresponds to $y_1 = 5.1$ on the horizontal axis. For $y_1 = 5.1$ of y_t, the value of y_{t+1} is $y_2 = 2.8$. If we continue like this, we obtain the complete oscillation and divergence (from the steady-state value 4) depicted by the movements of the arrows in Panel (B) of the figure. Notice that we will obtain a similar result if we choose an initial value above $y_s = 4$ (that is, $y_0 > y_s = 4$).

9.2.6 First-order, linear difference equations with variable coefficient

So far were concerned with the first-order, linear, autonomous difference equations in which both the coefficient (a) and the term (b) were assumed to be constants. Let us now consider the solution to the first-order, linear difference equations that are non-autonomous or that have variable coefficients; coefficients that are some functions of time.

Consider the first-order, linear, non-autonomous difference equation $y_{t+1} = a_t y_t + b$. In this equation the coefficient a_t is a function of time. How do we solve equations of this type? As we solved the first-order, linear, autonomous difference equations so far in this chapter, we can solve this equation through the iterative procedure. Suppose that the initial value of y, y_0, is known. Then we can write $y_{t+1} = a_t y_t + b$ as $y_1 = a_0 y_0 + b$, $y_2 = a_1 y_1 + b \quad = a_1[a_0 y_0 + b] + b =,\quad a_0 a_1 y_0 + a_1 b + b$, and $y_3 = a_2 y_2 + b = a_2[a_0 a_1 y_0 + a_1 b + b] + b$ $= a_0 a_1 a_2 y_0 + a_1 a_2 b + a_2 b + b$. If we continue like this for t times, we will obtain $y_t = [a_0 a_1 a_2 \ldots a_{t-2} a_{t-1}]y_0 + [a_1 a_2 \ldots a_{t-2} a_{t-1}]b \;+ [a_2 \ldots a_{t-2} a_{t-1}]b + \ldots + b$, which can be written, using the product notation introduced in Section 1.11, as

$$y_t = \prod_{i=0}^{t-1} a_i y_0 + \prod_{i=1}^{t-1} a_i b + \prod_{i=2}^{t-1} a_i b + \ldots + b = \prod_{i=0}^{t-1} a_i y_0 + b\left[\prod_{i=0}^{t-1}\frac{a_i}{a_0} + \prod_{i=1}^{t-1}\frac{a_i}{a_1} + \ldots + \prod_{i=k}^{t-1}\frac{a_i}{a_k} + \ldots + \prod_{i=t-1}^{t-1}\frac{a_i}{a_{t-1}}\right]$$
$$= \prod_{i=0}^{t-1} a_i y_0 + b\sum_{k=0}^{t-1}\prod_{i=k}^{t-1}\frac{a_i}{a_k}, \qquad (9.2.21)$$

which gives the solution to the first-order, linear difference equation with the variable coefficient a_t.

As an example, consider the first-order, linear difference equation $y_{t+1} = t y_t + 2$. Notice that the coefficient in this difference equation is a variable [that is, $a = f(t) = t$] and the term is a constant ($b = 2$). Suppose that the initial value of y, y_0, is known. Therefore, applying equation (9.2.21) we obtain the solution $y_t = \prod_{i=0}^{t-1} a_i y_0 + b\sum_{k=0}^{t-1}\prod_{i=k}^{t-1}\frac{a_i}{a_k} = \prod_{i=0}^{t-1} i y_0 + 2\sum_{k=0}^{t-1}\prod_{i=k}^{t-1}\frac{i}{k}$. Since $\prod_{i=0}^{t-1} i y_0 = 0$, the last equation reduces to $y_t = \prod_{i=0}^{t-1} a_i y_0 + b\sum_{k=0}^{t-1}\prod_{i=k}^{t-1}\frac{a_i}{a_k} = 2\sum_{k=0}^{t-1}\prod_{i=k}^{t-1}\frac{i}{k}$.

9.2.7 First-order, linear difference equations with variable term

In the last section we found how we could solve a first-order, linear difference equation with a variable coefficient if the initial value is known. Let us now attempt to find the solution to a first-order, linear difference equation with a variable term. If the term (b) is a variable, it will be some function of time.

Consider the first-order, linear, non-autonomous difference equation $y_{t+1} = ay_t + b_t$. In this equation the term b_t is a function of time. How do we solve difference equations of this type? As earlier, we can solve this equation through the iterative procedure. Suppose that the initial value of y, y_0, is known. Then we can write $y_{t+1} = ay_t + b_t$ as $y_1 = ay_0 + b_0$, $y_2 = ay_1 + b_1 = a[ay_0 + b_0] + b_1 = a^2 y_0 + ab_0 + b_1$, and $y_3 = ay_2 + b_2 = a[a^2 y_0 + ab_0 + b_1] + b_2 = a^3 y_0 + a^2 b_0 + ab_1 + b_2$. If we continue like this for t times, we will obtain $y_t = a^t y_0 + a^{t-1} b_0 + a^{t-2} b_1 + ... + b_{t-1}$, which can be written, using the sum notation introduced in Section 1.11, as

$$y_t = a^t y_0 + \sum_{k=0}^{t-1} b_k a^{t-1-k} , \tag{9.2.22}$$

which gives the solution to the first-order, linear difference equation with the variable term b_t .

As an example, consider the first-order, linear difference equation $y_{t+1} = 2y_t + t$. Notice that the term in this equation is a variable [that is, $b = f(t) = t$] and the coefficient is constant ($a = 2$). Therefore, applying equation (9.2.22) we obtain the solution as $y_t = a^t y_0 + \sum_{k=0}^{t-1} b_k a^{t-1-k} = 2^t y_0 + \sum_{k=0}^{t-1} k 2^{t-1-k}$.

9.2.8 First-order, linear difference equations with variable term and coefficient

So far we were attempting to find the solutions to the first-order, linear difference equations in which either the coefficient, or the term, or both were constants. Let us now attempt to find the solution to a first-order, linear difference equation in which both the coefficient and the term are variables. We can again use the now familiar iterative method.

Suppose that our equation in which both the coefficient and the term are variables is given by $y_{t+1} = a_t y_t + b_t$. As earlier, from the last equation we obtain $y_1 = a_0 y_0 + b_0$, $y_2 = a_1 y_1 + b_1 = a_1[a_0 y_0 + b_0] + b_1 = a_0 a_1 y_0 + a_1 b_0 + b_1$, and $y_3 = a_2 y_2 + b_2 = a_2[a_0 a_1 y_0 + a_1 b_0 + b_1] + b_2 = a_0 a_1 a_2 y_0 + a_1 a_2 b_0 + a_2 b_1 + b_2$. If we continue like this for t times, we will obtain $y_t = [a_0 a_1 a_2 ... a_{t-1}] y_0 + [a_1 a_2 ... a_{t-1}] b_0 + [a_2 ... a_{t-1}] b_1 + ... + a_{t-1} b_{t-2} + b_{t-1}$, which can be written as

$$y_t = y_0 \prod_{i=0}^{t-1} a_i + b_0 \prod_{i=0}^{t-1} \frac{a_i}{a_0} + b_1 \prod_{i=1}^{t-1} \frac{a_i}{a_1} + ... + b_k \prod_{i=k}^{t-1} \frac{a_i}{a_k} + ... + b_{t-1} \prod_{i=t-1}^{t-1} \frac{a_i}{a_{t-1}}$$,or as

$$y_t = y_0 \prod_{i=0}^{t-1} a_i + \sum_{k=0}^{t-1} b_k \prod_{i=k}^{t-1} \frac{a_i}{a_k} . \tag{9.2.23}$$

which gives the solution to the first-order, linear difference equation with the variable coefficient a_t and variable term b_t .

Suppose now that both the coefficient and the term of an equation are constants. Then the equation can be written as $y_{t+1} = ay_t + b$, which is identical with equation (9.2.1). Then the solution to equation is given by $y_t = y_0 \prod_{i=0}^{t-1} a_i + b \sum_{k=0}^{t-1} \prod_{i=k}^{t-1} \frac{a_i}{a_k}$, which is a special case of equation (9.2.23) when a and b are constants. Then the last equation can be written as $y_t = y_0 a^t + b\sum_{k=0}^{t-1} a^{t-1-k}$ or as $y_t = y_0 a^t + b[1 + a + a^2 + ... + a^{t-2} + a^{t-1}]$. Notice that the terms inside the brackets of the last equation constitute a geometric series with $k = a$ as the common ratio and $s_1 = 1$ as the first term. Therefore, we can apply equation (1.10.7) to obtain $1 + a + a^2 + ... + a^{t-2} + a^{t-1} = (1 - a^t)/(1 - a)$. Substituting this result into $y_t = y_0 a^t + b \times [a^{t-1} + a^{t-2} + ... + a + 1]$, we obtain $y_t = a^t y_0 + b[(1 - a^t)/(1 - a)]$, which is identical with equation (9.2.10). Therefore, the results in equations (9.2.10), (9.2.21) and (9.2.22) are the three special cases of the result in equation (9.2.23).

As an example, consider the equation $y_{t+1} = ty_t + t$. Notice that both the coefficient ($a_t = t$) and the term ($b_t = t$) are variables in this equation. Therefore, applying equation (9.2.23) we obtain the solution as $y_t = y_0 \prod_{i=0}^{t-1} a_i + \sum_{k=0}^{t-1} b_k \prod_{i=k}^{t-1} \frac{a_i}{a_k} = \prod_{i=0}^{t-1} iy_0 + \sum_{k=0}^{t-1} k \prod_{i=k}^{t-1} \frac{i}{k}$. Since the first term on the RHS in the last equation is zero, the solution reduces to $y_t = y_0 \prod_{i=0}^{t-1} a_i + \sum_{k=0}^{t-1} b_k \prod_{i=k}^{t-1} \frac{a_i}{a_k} = \sum_{k=0}^{t-1} k \prod_{i=k}^{t-1} \frac{i}{k}$.

9.2.9 First-order, nonlinear difference equations

One of our main aims in the present chapter so far was to solve first-order linear difference equations. We were also concerned with the long-run behavior of the underlying variable(s). We solved these equations explicitly and attempted to understand the long-run behavior of the variables either by plotting the solutions or by constructing phase diagrams. Let us now consider first-order, nonlinear, autonomous difference equations. As a general case, consider equation (9.2.5). Assume for convenience $y = k; a = 1 \; and \; b = 0$. Then the equation becomes $y_{t+1} = y_t^{\alpha}$. Many of the first-order, nonlinear difference equations cannot be solved explicitly. Often, in the subjects of our interest, a qualitative analysis of convergence or divergence is sufficient, and such an analysis can be carried out using phase diagram introduced in Section 9.2.5.

We observed in Section 9.2.5 that, in the case of first-order, linear difference equations, if the slope of a phase line was in between 0 and +1 (-1 and 0), then the time path represented by the phase line would converge (converge with oscillation) to the steady-state; and if the slope was greater than 1 (less than -1), then the time path would diverge (diverge with oscillation) from the steady-state. Therefore, the absolute value of the slope of the phase line determined convergence or divergence and its sign determined oscillation or non-oscillation. This result is applicable in the case of the phase diagrams of nonlinear difference equations too but with a slight modification. Notice that the slope is the same at every point on a linear phase line and, therefore, it does not matter where on the phase line we measure it. But, the slope is different at different points on a nonlinear phase line. Therefore, the modification mentioned is that the slope in the case of a nonlinear difference equation is to be evaluated at the equilibrium or the steady-state point. This means that we can apply the above result with the slope of the phase line evaluated at the steady-state.

Notice that we can obtain the same result as above through **linearization of nonlinear difference equation**. Let us linearize the nonlinear difference equation $y_{t+1} = f(y_t) = y_t^{\alpha}$. We know that in steady-state the value of $y_{t+1} = y_t$ is y_s. We can now expand this function around y_s using Taylor series introduced in Section 3.6 to obtain $f(y_{t+1}) = f(y_s) + f'(y_s)(y_{t+1} - y_s)$, where we have omitted the remainder term for convenience. Notice that the term $(y_{t+1} - y_s)$ in the last equation is just a constant at any particular t. Therefore, whether $f(y_{t+1}) = f(y_s)$, that is whether $f(y_{t+1})$ converges to $f(y_s)$, is determined solely by $f'(y_s)$. But, $f'(y_s)$ is the slope of the phase line evaluated at the steady-state. Therefore, we have the same result as that above.

As an example, consider the above difference equation with $\alpha = 0.5, \alpha = -0.5, \alpha = 3, \; and \; \alpha = -1.5$; that is, $y_{t+1} = f^1(y_t) = y_t^{0.5}$, $y_{t+1} = f^2(y_t) = y_t^{-0.5}$, $y_{t+1} = f^3(y_t) = y_t^3$, and $y_{t+1} = f^4(y_t) = y_t^{-1.5}$. Following the procedure outlined in Section 9.2.5, we can construct the phase diagrams of these four first-order, nonlinear, autonomous difference equations as illustrated in Panels (A) through (D) of Figure 9.2.7.

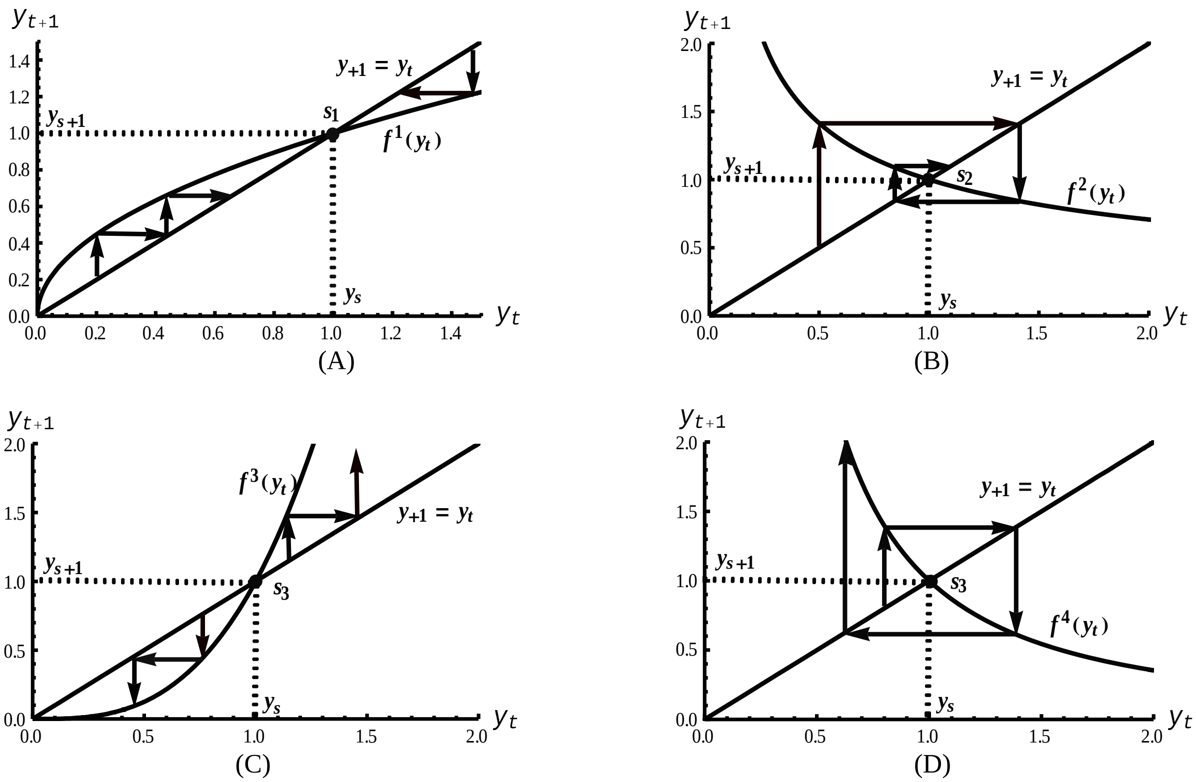


Figure 9.2.7

Before we check whether the phase diagrams in Figure 9.2.7 conform to our above definition of convergence and divergence, let us derive the steady-state values of the above four equations. We know that the steady-state of a particular difference equation occurs when the phase line of the equation intersects the 45º line or the line of $f(y_t) = y_{t+1}(= y_t)$. Consider first equation $y_{t+1} = f^1(y_t) = y_t^{0.5}$, the phase line of which is illustrated in Panel (A) of Figure 9.2.7. This phase line intersects the 45º line at point s_1. At this point we have $y_{t+1} = y_t^{0.5}$ or $y_t - y_t^{0.5} = 0$. Notice that at this point $y_t = y_s$, which can be substituted into the last equation to obtain $y_s - y_s^{0.5} = 0$ and solved to obtain the two roots $y_s = 0$ *and* $y_s = 1$. Therefore, we have two steady-states (or **multiple steady-states** or **multiple equilibria**); one occurs when $y_t = y_s = 0$ and the other occurs when $y_t = y_s = 1$. Notice also that these are the steady-state values of all the remaining difference equations and that we have $y_s = 1$ ($y_s = 0$) is (not) shown as the steady-state values in all the four phase diagrams in Figure 9.2.7.

We can now check whether the phase diagrams in Figure 9.2.7 conform to our definitions of convergence and divergence. Consider first the phase diagram in Panel (A) of the figure. Notice that the slope of the phase line $y_{t+1} = f^1(y_t) = y_t^{0.5}$ is $f^{1\prime}(y_t) = 0.5y_t^{-0.5}$. When this slope is evaluated at $y_s = 1$, we obtain $f^{1\prime}(y_s) = 0.5y_s^{-0.5} = 0.5(1)^{-0.5} = 0.5$. Since the slope of the phase line at the steady-state lies in between 0 and 1 [that is, $0 < f^{1\prime}(y_s) = 0.5 < 1$], the time path of y_t converges to the steady-state value $y_s = 1$. Therefore, the time path of y_t represented by the phase line $f^1(y_t)$ is in conformity with our definition of convergence. Since $f^{1\prime\prime}(y_t) = (-0.5)(0.5)y_t^{-0.5} < 0$ for all $y_t > 0$, the phase line crosses the 45º line only at the point s_1, except at 0. This implies that y_t converges to $y_s = 1$ for all initial values larger than $y_t > 0$ and, therefore, this equilibrium is a **globally stable equilibrium**. Once y_t reaches $y_s = 1$, it remains there unless there is an external shock.

Consider now the phase line $f^2(y_t)$ in Panel (B). The slope of this phase line is $f^{2\prime}(y_t) = -0.5y_t^{-1.5}$ and when this slope is evaluated at $y_s = 1$ we obtain $f^{2\prime}(1) = -0.5(1)^{-1.5} = -0.5$. Since the slope of the phase line $f^2(y_t)$ lies in between -1 and 0 (that is, $-1 < f^{2\prime}(1) = -0.5 < 0$), y_t converges to $y_s = 1$ but with oscillation.

This is in conformity with our definition of convergence with oscillation. Since $f^{2}{''}(y_t) = (-0.5)(-1.5)y_t^{-0.5} > 0$ for all $y_t > 0$, the phase line will cross the 45º line only at point s_2. This implies that y_t converges to $y_s = 1$ for initial values larger than $y_t > 0$ and, therefore, this equilibrium is also a **globally stable equilibrium**.

The slope of the phase line $f^3(y_t)$ in Panel (C) at $y_s = 1$ is $f^{3}{'}(y_s) = 3y_s^2 = 3\times 1^3 = 3$. Since this slope is greater than 1 [that is, $f^{3}{'}(y_s) = 3 > 1$], y_t diverges in both directions from the steady-state value $y_s = 1$. This is also in conformity with our definition of divergence. Since $f^{3}{''}(y_t) = 3\times 2\times y_t^{-0.5} > 0$ for all $0 < y_t < +\infty$, the phase line will cross the 45° line only at the point s_3. Therefore, if we choose the initial value such that $0 < y_0 < y_s$, then y_t converges to 0. This implies that the equilibrium at point $y_s = 0$ is a **locally stable equilibrium** and the equilibrium at point $y_s = 1$ is an **unstable equilibrium**.

Finally, the slope of the phase line $f^4(y_t)$ in Panel (D) at $y_s = 1$ is $f^{4}{'}(y_s) = -1.5y_s^{-2.5} = -1.5(1)^{-2.5} < -1$. Since this slope is less than -1 [that is, $f^{4}{'}(y_s) = -1.5(1)^{-2.5} < -1$], y_t diverges from the steady-state value $y_s = 1$ but with oscillation. This, again, is in conformity with our definition of divergence. Since $f^{4}{''}(y_t) = -1.5\times -2.5y_t^{-3.5} > 0$ for all $0 < y_t < +\infty$, the phase line will cross the 45^0 line only at point s_4. This implies that the equilibrium at point $y_s = 1$ is an unstable equilibrium.

9.2.10 Application examples

Example 1: Suppose that the aggregate consumption at time t in an economy is given by $C_t = C_0 + cY_{t-1}$, where $C_0, c, and\ Y$ denote the autonomous consumption, the marginal propensity to consume (MPC), and the national income, respectively. Also suppose that the total national income in period t in the economy is given by $Y_t = C_t + I_t$, where I_t denotes the total investment in the economy, which is assumed to be equal to a constant I_0. Determine the time path of the economy's national income and determine whether this path converges or diverges using a phase diagram if $C_0 = 40, I_0 = 10, Y_0 = 300,\ and\ c = 0.8$. All values are in billions of dollars.

Solution: This is a **discrete time income determination** problem. Substituting the given values into the equations we obtain $C_t = 40 + 0.8Y_{t-1}, Y_t = C_t + 10$ and $Y_{t+1} = 40 + 0.8Y_t + 10 = 50 + 0.8Y_t$. The last equation is identical with difference equation (9.2.1) for $a = 0.8\ and\ b = 50$. The particular solution of this equation is given by equation (9.2.17): $Y_s = b/(1-a) = 50/(1-0.8) = \$250$ billion. Therefore, applying equation (9.2.18) and treating $C = Y_0$, we obtain the path of Y_t as the sum of $Y_c\ and\ Y_s : Y_t = [C_0 - b/(1-a)]a^t + b/(1-a)$ $= [300 - 50/(1-0.8)](0.8)^t + 50/(1-0.8) = 50(0.8)^t + 250$, which is graphed in Panel (A) of Figure 9.2.8. The phase diagram for the difference equation $Y_{t+1} = 50 + 0.8Y_t$ is illustrated in Panel (B) of the same figure. The slope of the phase line is $f'(Y_t) = 0.8$. Since the slope being $0 < 0.8 < 1$, our result in Section 9.2.5 suggests that Y_t converges to the steady-state. The steady-state value of Y_t can be found by equating the difference equation with the equation for the 45º line. Therefore, the steady-state value of Y_t is $Y_{t+1} = 50 + 0.8Y_t = Y_t$, or $50 + 0.8Y_t = Y_t$, or $Y_t = Y_s = 250$, which is shown in Panel (B) of the figure. Since the phase line is a straight line, it will cross the 45^0 line at only one point; that is, only at the steady-state point $Y_s = 250$. This implies that the equilibrium or the steady-state is stable. Therefore, the economy's long-run stable equilibrium income is \$250 billion.

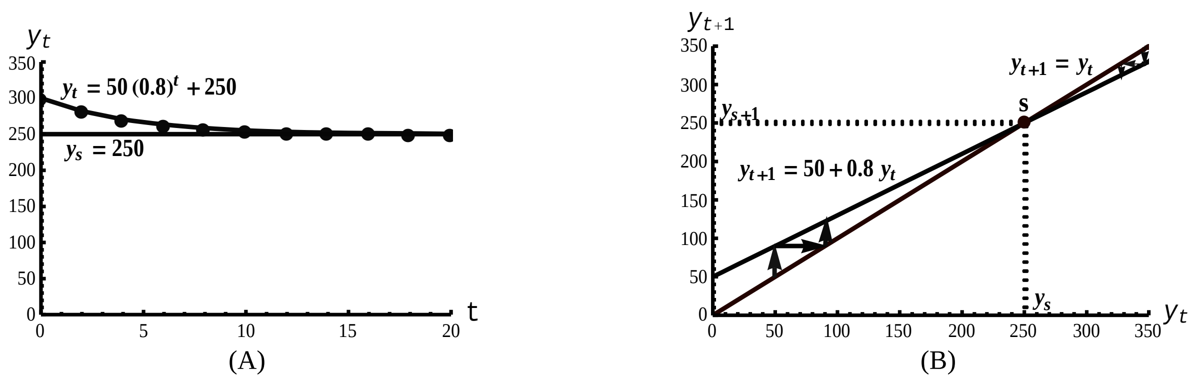


Figure 9.2.8

Example 2: Suppose that the demand for a good at time t is given by the function $q_{dt} = 15 - 2p_t$, where q_{dt} represents the quantity demanded of the good at time t and p_t represents the price of the good at time t; and the supply of the good is given by the function $q_{st} = 5 + p_{t-1}$, where q_{st} represents the quantity supplied of the good at time t and p_{t-1} represents the price of the good at time $t-1$. Does the path of price p_t converge to the equilibrium level of price p_s (if it exists) and, if so, what is the equilibrium price?

Solution: This is an example of **cobweb model** of **price determination**. The equilibrium in the market for the good occurs when $q_{dt} = 5 - 2p_t$ is equal to $q_{st} = 15 + p_{t-1}$. Assuming the market for the good attains equilibrium, we can write $q_{dt} = q_{st}$, which implies that $15 - 2p_t = 5 + p_{t-1}$ or $p_t = 5 - 0.5p_{t-1}$, which is identical with equation (9.2.1) for $y = p, a = -0.5, and\ b = 5$. Therefore, applying equation (9.2.17), we obtain the particular solution $p_s = b/(1-a) = 5/1.5 = 3.33$. The complementary function is given by $p_c = Ca^t = C(-0.5)^t$. Therefore, treating $p_0 = C$ and applying equation (9.2.18), we obtain the complete solution $p_t = [p_0 - 5/\{1-(-0.5)\}](-0.5)^t \ + 5/\{1-(-0.5)\} = [p_0 - 3.33](-0.5)^t + 3.33$. Assuming $p_0 = 6$, the last equation can be written as $y_t = 2.67(-0.5)^t + 3.33$, the graph of which is illustrated in Panel (A) of Figure 9.2.9. The phase diagram for the difference equation $p_{t+1} = 5 - 0.5p_t$ is illustrated in Panel (B) of the figure. The slope of the phase line is $f'(p_t) = -0.5$. Since the slope is $-1 < -0.5 < 0$, the result in Section 9.2.5 suggests that p_t must converge to the steady-state with oscillation. The steady state value of p_t can be found by equating the difference equation with the equation for the 45° line. Therefore, the steady-state value of p_t is $p_{t+1} = 5 - 0.5p_t = p_t$, or $5 - 0.5p_t = p_t$, or $p_t = p_s = 3.33$, which is illustrated in Panel (B) of the same figure. Since the phase line is a straight line, it will cross the 45º line at only one point; that is, only at the steady-state point $p_s = 3.33$. This implies that the equilibrium is stable.

Example 3: Suppose that Ms. Janet purchased the share of a company for \$100. She expects the value of the share to increase by 5 per cent every month. What will be the value of the share after 24 months?

Solution: The value of the share after one month (F_1) is equal to the value of the share at the time of the purchase of the share (F_0) plus the increased value, which can be written as $F_1 = F_0 + 0.05F_0 = (1.05)F_0$. Using the notation t to represent time, the last equation can be written as $F_t = (1.05)F_{t-1}$ or $F_{t+1} = (1.05)F_t$, which is a first-order, linear, autonomous, homogenous difference equation and is a special case of equation (9.2.1) with $a = 1.05$ and $b = 0$. To find the particular solution we can use equation (9.2.17): $F_s = b/(1-a) = 0/(1-1.05) = 0$ because $b = 0$. The complementary function, as before, is $F_c = Ca^t = C(1.05)^t$. Therefore, using equation (9.2.18),

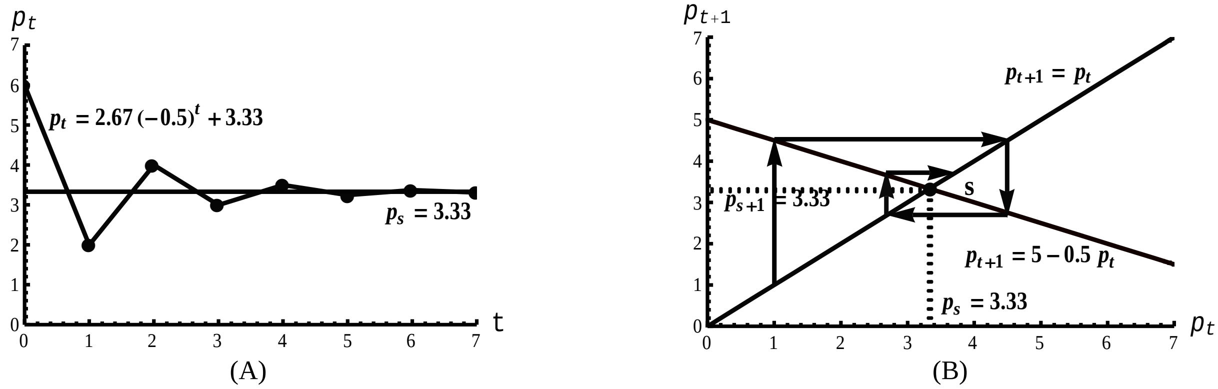
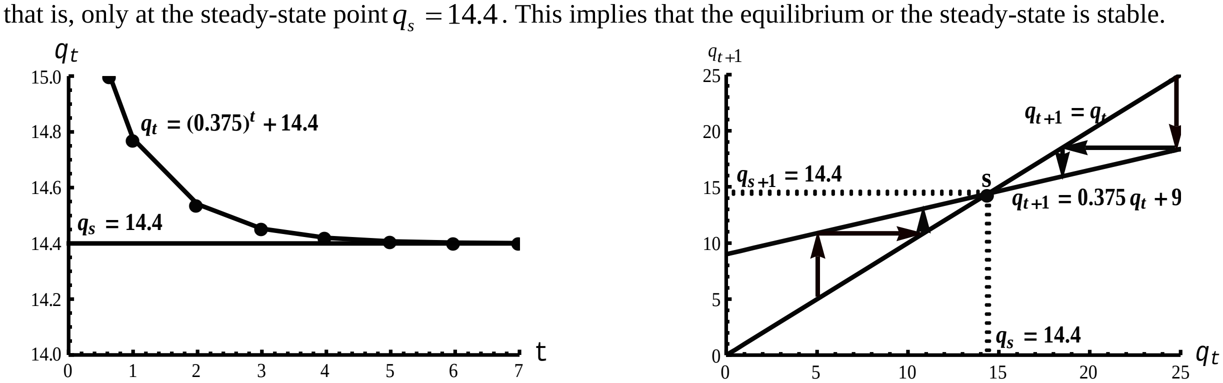


Figure 9.2.9

we obtain the complete solution to the difference equation: $F_t = F_s + F_c = y_0 a^t + b[(1-a^t)/(1-a)] = y_0 a^t$. Since $a = 1.05$, the solution can be written as $F_t = a^t F_0 = F_0(1.05)^t$. Now treating $F_0 = P$, the solution becomes $F_t = P(1.05)^t = P(1+0.05)^t$, which is identical with equation (9.4.27) for $r = 0.05$. Therefore, the value of the share after 24 months will be $F_{24} = P(1.05)^t = 100(1.05)^{24} = \$322.51$.

Example 4: Suppose that the quantities supplied and demanded of a good at time t are given by $q_{st} = 6 + 3p_t$ and $q_{dt} = 10 - p_t + 0.5q_{t-1}$, respectively, where p_t denotes price per unit of the good. Assuming that the market clears each period, derive the difference equation for quantity and solve it. Determine whether the path of quantity converges or diverges with or without oscillation.

Solution: This example is the case of **habit formation hypothesis** due to the presence of the lagged quantity demanded (q_{t-1}) in the demand function. Let us first solve the supply equation for p_t to obtain $p_t = (q_t - 6)/3$. We shall now substitute this into the demand equation to obtain $q_t = 10 - p_t + 0.5q_{t-1}$ $= 10 - [(q_t - 6)/3] + 0.5q_{t-1}$, which can be simplified to obtain $q_t = 0.375q_{t-1} + 9$ or $q_{t+1} = 0.375q_t + 9$. The last equation is the required difference equation for quantity. Notice that this equation is identical with equation (9.2.1) for $a = 0.375$ and $b = 9$. Therefore, after finding the particular solution using equation (9.2.17), finding the complementary function $q_c = Ca^t$, and combining them, we can apply equation (9.2.20) to obtain the complete solution $q_t = Aa^t + d = A(0.375)^t + [9/(1-0.375)] = (0.375)^t + 14.4$, where we have assumed $A = 1$. The path of q_t is illustrated in Panel (A) of Figure 9.2.10. The phase diagram for the difference equation $q_{t+1} = 0.375q_t + 9$ is illustrated in Panel (B) of the same figure. The slope of the phase line is $f'(q_t) = 0.375$. Since the slope is $0 < 0.375 < 1$, the result in Section 9.2.5 suggests that q_t converges to the steady-state without oscillation. The steady state value of q_t can be found by equating the difference equation with the equation for the 45º line. Therefore, the steady-state value of q_t is $q_{t+1} = 9 + 0.375q_t = q_t$, or $9 + 0.375q_t = q_t$, or $q_t = q_s = 14.4$, which is illustrated in Panel (B) of the figure. Since the phase line is a straight line, it will cross the 45° line at only one point; that is, only at the steady-state point $q_s = 14.4$. This implies that the equilibrium or the steady-state is stable.

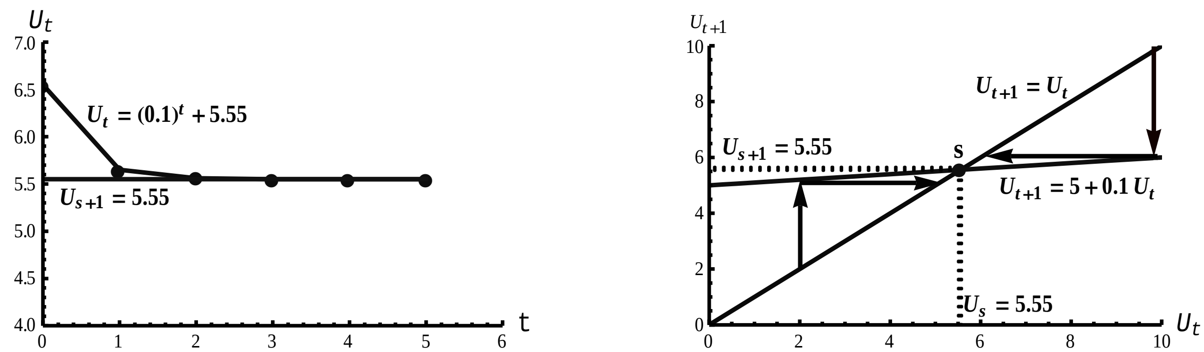


(A) (B)

Figure 9.2.10

Example 5: Assume that the rate of unemployment (U) in an economy in period t is given by the function $U_t = 5 + 0.1U_{t-1}$, where the values are in percentages. Solve this difference equation and determine whether unemployment converges to or diverges from the steady-state (if it exists) level or the **natural rate of unemployment**.

Solution: Notice that the given difference equation is identical with equation (9.2.1) for $a = 0.1$ and $b = 5$. Therefore, after finding the particular solution using equation (9.2.17), the complementary function $U_c = Ca^t$, and combining them, we can apply equation (9.2.18) to obtain the complete solution to the difference equation as $U_t = Ca^t + b/(1-a) = C(0.1)^t + 5/(1-0.1) \; = (0.1)^t + 5.55$, where we have assumed $C = 1$. The path of U_t is illustrated in Panel (A) of Figure 9.2.11. The phase diagram for the difference equation $U_t = 5 + 0.1U_{t-1}$ or $U_{t+1} = 5 + 0.1U_t$ is illustrated in Panel (B) of the same figure. The slope of the phase line is $f'(U_t) = 0.1$. Since the slope is $0 < 0.1 < 1$, the result in Section 9.2.5 suggests that U_t converges to the steady-state or the natural rate of unemployment without oscillation. The steady state value of U_t can be found by equating the difference equation with the equation for the 45° line. Therefore, the steady-state value of U_t is $U_{t+1} = 5 + 0.1U_t = U_t$, or $5 + 0.1U_t = U_t$, or $U_t = U_s = 5.55$, which is shown in Panel (B) of the figure. Since the phase line is a straight line, it will cross the 45° line at only one point; that is, only at the steady-state point $U_s = 5.55$. This implies that the equilibrium or the steady-state is stable.



(A) (B)

Figure 9.2.11

Example 6: Suppose that the total saving at time t in an economy is given by $S_t = sY_t = 0.2Y_t$, where $s = 0.2$ denotes the marginal propensity to save (MPS) and Y_t denotes the national income of the economy. Also suppose that the total investment in the economy is given by $I_t = m[Y_t - Y_{t-1}] = 2.2[Y_t - Y_{t-1}]$, where $m = 2.2$ denotes the **capital-output ratio** in the economy. Setup the difference equation for national income and determine whether the economy's national income converges to or diverges from the steady-state (if it exists).

Solution: This is an example of the discrete time version of **Harrod's model** of **economic growth**. At the equilibrium or the steady-state level of income we will have $S_t = I_t$, which implies that $0.2Y_t = 2.2[Y_t - Y_{t-1}]$ or, after simplification, $Y_{t+1} = 1.1Y_t$, which is identical with equation (9.2.1) for $a = 1.1$ and $b = 0$. Therefore, we can apply equation (9.2.11) to obtain the solution $Y_t = a^tY_0 = (1.1)^tY_0$. Assuming $Y_0 = \$100$ billion, then the last equation can be written as $Y_t = 100(1.1)^t$, which is graphed in Panel (A) of Figure 9.2.12. The phase diagram for the difference equation $Y_{t+1} = 1.1Y_t$ is illustrated in Panel (B) of the same figure. The slope of the phase line is $f'(Y_t) = 1.1$. Since the slope is $1.1 > 1$, the result in Section 9.2.5 suggests that Y_t diverges from the steady-state without oscillation. The steady state value of Y_t can be found by equating the difference equation with the equation for the 45° line. Therefore, the steady-state value of Y_t is $Y_{t+1} = 1.1Y_t = Y_t$, or $1.1Y_t = Y_t$, or $Y_t = Y_s = 0$, which is shown in Panel (B) of the figure.

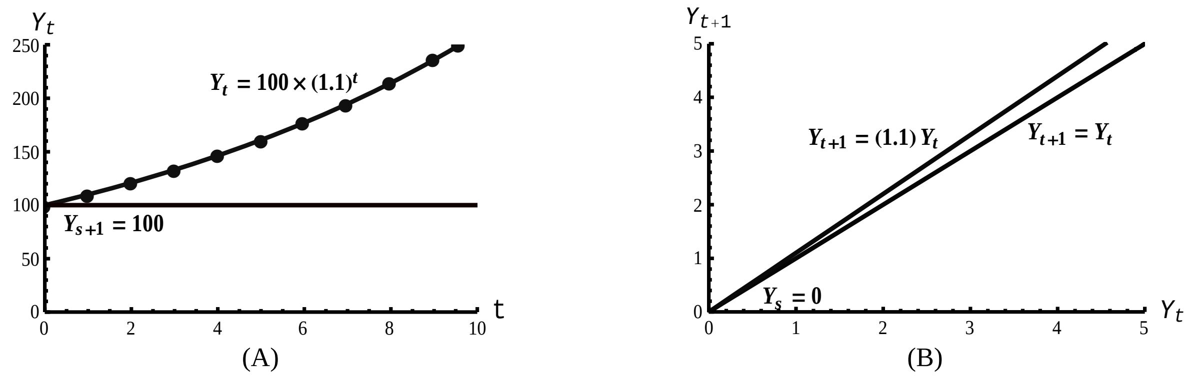


Figure 9.2.12

Example 7: Suppose that the total output (Y_t) at time t in an economy is produced, using K_t units of capital, L_t units of labor and using technology A_t, with the Cobb-Douglas production function $Y_t = F(K_t, L_t) = K_t^{\alpha}(A_t L_t)^{1-\alpha}$, where $0 < \alpha < 1$. Then output per unit of **effective labor** ($A_t L_t$) is given by $y_t = f(k_t) = Y_t / A_t L_t = K_t^{\alpha} / (A_t L_t)^{\alpha} = k_t^{\alpha}$. Suppose also that the capital stock in the economy changes by $K_{t+1} - (1-\delta)K_t = sY_t$, where s *and* δ denote the constant **saving rate** and **depreciation rate** in the economy, respectively, and that the growth rate of effective labor is $g + n$ [that is, $(A_{t+1}L_{t+1} - A_t L_t) / A_t L_t = g + n$]. Lastly, assume that $0 < s < 1, 0 < \delta < 1, g > 0,$ *and* $n > 0$. Derive the difference equation in k and determine whether the time path of k_t converges to or diverges from the steady-state level (if it exists).

Solution: This is an example of the discrete timer version of **Solow model** of economic growth, with positive growths in labor and technology. Let us first derive the required difference equation. Notice that $sY_t = K_{t+1} - (1-\delta)K_t$, or $sY_t / A_t L_t = [K_{t+1} / A_t L_t] - (1-\delta)(K_t / A_t L_t) =$ $[K_{t+1} / A_{t+1}L_{t+1}][A_{t+1}L_{t+1} / A_t L_t]$ $-(1-\delta)(K_t / A_t L_t)$. Notice also that $[A_{t+1}L_{t+1} - A_t L_t] / A_t L_t = n + g$, which implies that $A_{t+1}L_{t+1} / A_t L_t = 1 + g + n$. Therefore, the last equation can be written as $sy_t = [K_{t+1} / A_{t+1}L_{t+1}][A_{t+1}L_{t+1} / A_t L_t] - (1-\delta)(K_t / A_t L_t)$ $= k_{t+1}(1 + g + n) - (1-\delta)k_t$, or $sy_t = k_{t+1}(1+g+n) - (1-\delta)k_t$. Since $y_t = k_t^{\alpha}$, we can write the last equation as $sk_t^{\alpha} = k_{t+1}(1 + g + n) - (1-\delta)k_t$, or as $k_{t+1} = [(1-\delta)k_t + sk_t^{\alpha}]/(1 + g + n) = [(1-\delta)k_t / (1 + g + n)]$ $+ [sk_t^{\alpha} / (1 + g + n)]$. Since this is a first-order, nonlinear difference equation, it is difficult to be solved explicitly. Instead, we can use the phase diagram of the equation to present a qualitative analysis.

Before we plot the phase line of the above difference equation [$k_{t+1} = h(k_t)$], we need to specify its curvature. The curvature of the phase line (or of any other curve or line) depends on the first and the second derivatives of the function that represents the phase line. The first derivative of the phase line is $h'(k_t) = [(1-\delta)/(1 + g + n)] + [s\alpha k_t^{\alpha-1} / (1 + g + n)]$. This implies that, since $0 < \alpha < 1, 0 < \delta < 1, 0 < s < 1, g > 0,$ *and* $n > 0$, $h'(k_t) > 0$ and, therefore, the phase line increases. The second derivative of the phase line is $h''(k_t) = [s(\alpha - 1)\alpha k_t^{\alpha-2} / (1 + g + n)]$. This implies that, since $0 < \alpha < 1, 0 < s < 1, g > 0,$ *and* $n > 0$, $h''(k_t) < 0$. Therefore, the phase line increases at a diminishing rate. In other words, the phase line is strictly concave. Since $k_{t+1} = 0$ when $k_t = 0$, the phase line passes through the origin. Therefore, the phase line will intersect the 45° line at two points only: (0, 0) and $k_{t+1} = k_t$. At $k_{t+1} = k_t$, we have $[(1-\delta)k / (1 + g + n)] + [sk_t^{\alpha} / (1 + g + n)] = k_t$, which simplifies to $k_s = [(g + n + \delta)/s]^{1/(\alpha-1)}$. Therefore, the two steady-state values of k are $k_s = 0$ and $k_s = [(g + n + \delta)/s]^{1/(\alpha-1)}$. Since $h''(k_t) < 0$ for all $k > 0$, the phase line intersects the 45^0 line only at (except at 0) the steady-state point s giving $k_s = [(g + n + \delta)/s]^{1/(\alpha-1)}$. This implies that the local steady state $k_s = 0$ is unstable, and the local steady-state $k_s = [(g+n+\delta)/s]^{1/(\alpha-1)}$ is stable, and it is a global steady-state too. If we substitute $k_s^{\alpha-1} = [(g+n+\delta)/s]$ into the first derivative of the difference equation, $h'(k_t)$, we obtain $h'(k_s) = [(1-\delta)/(1+g+n)] + [\alpha(g+n+\delta)/(1+g+n)] < 1$. In other words, we have the inequality $0 < h'(k_s) < 1$. Therefore, using our results in Section 9.2.9, we can infer that k converges without oscillation to the steady-state level $k_s = [(g+n+\delta)/s]^{1/(\alpha-1)}$. We can now conclude that capital per effective labor in the economy converges to $k_s = [(g+n+\delta)/s]^{1/(\alpha-1)}$ without oscillation. All these are illustrated in Figure 9.2.13.

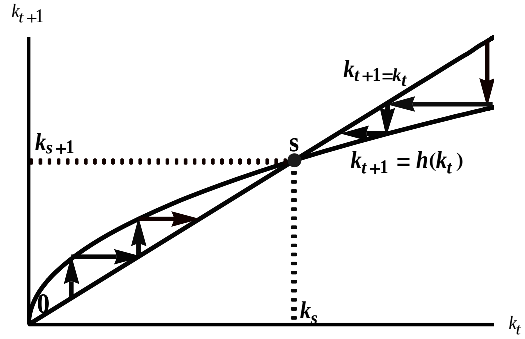


Figure 9.2.13

9.2.11 Exercises

1. Solve the following difference equations. Definitize and verify the solutions using $y_0 = 1$ *and* $y_1 = 2$ when $t = 0$ *and* $t = 1$, respectively.

(i) $y_{t+1} + 4y_t = 0$ (ii) $y_{t+1} + 4 + 4y_t = 4$ (iii) $y_{t+1} - 4y_t = 0$ (iv) $y_{t+1} - 4 - 4y_t = 4$
(v) $y_{t+1} = (1/4)y_t + 2$ (vi) $y_{t+1} = (-1/4)y_t + 2$ (vii) $y_{t+1} = y_t + 2$ (viii) $y_{t+1} = y_t$.

2. Determine whether y_t converges to or diverges from steady-state (if any) with or without oscillation by sketching the graphs of the solutions to and the phase diagrams of the difference equations in exercise 1 above.

3. Solve the following difference equations assuming $y_0 = 1$.

(i) $y_{t+1} = 2^t y_t$ (ii) $y_{t+1} = y_t + 2^t$ (iii) $y_{t+1} = 2y_t + 2^t$ (iv) $y_{t+1} = 2y_t + 2$
(v) $y_{t+1} = 1^t y_t + 2^t$. (vi) $y_{t+1} = y_t + t$ (vii) $y_{t+1} = y_t + 2t$.

4. Find the steady-states (if any) of the following difference equations and determine whether y_t converges to (or diverges from) the steady-states using phase diagrams.

(i) $y_{t+1} = 2 + y_t^2$ (ii) $y_{t+1} = 2 - y_t^2$ (iii) $y_{t+1} = 2 + (1/y_t)$ (iv) $y_{t+1} = 2 - (1/y_t)$.

4. **Application exercise**. Suppose that the supply of and demand for a good are given by $q_{st} = 10 + 4p_t + q_{t-1}$ and $q_{dt} = 20 - 2p_t$, respectively, where q_{st} denotes the quantity supplied, q_{dt} denotes the quantity demanded, and p_t denotes the price per unit of the good. Derive the difference equation for q_t and solve it assuming $q_0 = 1$. Also suppose that the market for the good clears each period. Sketch the graph of the path of q_t and draw the phase diagram of the difference equation. Does q_t converge to or diverge from the steady-state, if any, with or without oscillation?

5. **Application exercise**. Suppose that the supply of and demand for a good are given by $q_{st} = 10 + 4p_t$ and $q_{dt} = 20 - 2p_t + 0.5q_{t-1}$, respectively, where q_{st} denotes the quantity of the good

supplied, q_{dt} denotes the quantity demanded, and p_t denotes the price per unit of the good. Derive the difference equation for q_t and solve it assuming $q_0 = 1$. Also suppose that the market for the good clears each period. Sketch the graph of the path of q_t and draw the phase diagram of the difference equation. Does q_t converge to or diverge from the steady-state, if any, with or without oscillation?

6. **Application exercise**. Suppose that the aggregate consumption in an economy is given by $C_t = C_0 + cY_{t-1}$, where C_0, c, and Y_{t-1} denote autonomous consumption, marginal propensity to consume, and national income, respectively. Also suppose that the total national income in the economy is given by $Y_t = C_t + I_t$, where I_t denotes the total investment in the economy which is assumed to be equal to a constant I_0. Determine the time path of the economy's national income and determine whether this path converges or diverges using a phase diagram if $C_0 = 60, I_0 = 30, Y_0 = 500,\ and\ c = 0.75$. All values are in billions of dollars.

7. **Application exercise**. Assume that an individual deposits \$1000 in her bank account to day. What will the amount in her account after 10 years if the interest rate is 0.08 compounded annually?

8. **Application exercise**. Assume that the quantities demanded and supplied of a good are given by $q_{dt} = 10 - 0.5p_t$ and $q_{st} = 2 - 0.1p_{t-1}$, respectively, where p_t denotes the price. Assume also that the market clears each period and that $p_0 = 20$. Derive the difference equation for price and solve it. Determine, with the help of a phase diagram, whether the path of the underlying variable converges to or diverge from the steady-state (if any) with or without oscillation.

9. **Application exercise**. Suppose that the quantity demanded and supplied of a good are given by $q_{dt} = 50 - p_t$ and $q_{st} = 10 + 0.2p_t$, respectively, where p_t denotes price. Also suppose that $p_{t+1} = p_t - 0.1[q_{st} - q_{dt}]$ and that $p_0 = 55$. Derive the difference equation for price and solve it. Using a phase diagram, determine whether the time path of the underlying variable converges to or diverges from the steady-state (if any) with or without oscillation.

9.2.12 Mathematica applications

Mathematica's built-in-command **RSolve** can be used to solve difference equations or **recurrence equations**. The command **RSolve[*eqn*,*y*[*t*],*t*]**solves a difference equation ***eqn*** for ***y*[*t*]**.

Suppose that we have a general first-order, linear, autonomous, non-homogenous difference equation as the one in equation (9.2.1): $y_{t+1} = ay_t + b$. The solution to this difference equation can be obtained using the command **RSolve** as

RSolve[y[t+1]==a×y[t]+b,y[t],t]
{{y[t]→ -(((1-a^t) b)/(-1+a))+a^{-1+t} C[1]}} ,

which is identical with the general solution in equation (9.2.10) for y_0 = C[1]. If we use the ***initial condition*** or the ***initial value*** $y_0 = 0$, then the general solution can be obtained as

RSolve[{y[t+1]==a×y[t]+b,y[0]==0},y[t],t]
{{y[t]→ ((-1+a^t) b)/(-1+a)}}

As a specific example of the application of the command **RSolve**, consider the first difference equation we solved in Section 9.2.2: $y_{t+1} = 2y_t + 4$. This equation can be solved using the command **RSolve** as

RSolve[y[t+1]==2y[t]+4,y[t],t]
{{y[t]→ -4(1-2^t)+2^{-1+t}C[1]}}

Notice that this result is identical with the result we obtained in Section 9.2.2 with C[1]=1. The same difference equation can be solved with the initial condition $y_0 = 1$ to obtain

RSolve[{y[t+1]==2×y[t]+4,y[0]==1},y[t],t]

```
{{y[t]→-4+5 2^t}}
```

9.3 Higher-Order Linear Difference Equations

9.3.1 Introduction

In Section 9.2 we were concerned with difference equations that involved only the first difference or with the first-order difference equations. In other words, we were concerned with the relationship between y_{t+1} and y_t, or equivalently with y_t and y_{t-1}, or with Δy_t. It must be emphasized that difference equation are not restricted to first-order. They may, in fact, assume any order. One can cite a number of examples of higher order difference equations from the subjects of our interest. **Samuelson's multiplier-accelerator interaction model**, **long-run Philip's curve**, modified form of the cobweb model solved in example 2 of Section 9.2.10, Cournot duopoly model, etc. are just a few to cite. This point to the fact that the students of economics, business and finance need to learn the solution and other topics related to the second-and-higher-order difference equations. Therefore, we turn our attention to such equations in the present section.

9.3.2 Second-order, linear, autonomous difference equations

To begin with, consider the simplest possible second-order, linear, autonomous difference equation

$$y_{t+2} = a_1 y_{t+1} + a_2 y_t + b, \qquad (9.3.1)$$

which is identical with equation (9.2.3) for $y = v$. Our aim in this section is to find out the solution to the general equation (9.3.1). The method of solution to the first-order, linear, autonomous difference equations exposed in Section 9.2 can lead us in finding it. As in the case of the solution to equation (9.2.18), the complete solution to equation (9.3.1) also comprises of two parts: the complementary function and the particular solution. Therefore, let us first find out the particular solution before moving on to the determination of the complementary function.

Notice that the particular solution in the case of the solution to the general first-order, linear, autonomous difference equation $y_{t+1} = ay_t + b$ was given in equation (9.2.17): $y_s = b/(1-a)$. This was based on the assumption that $y_t = p$ and $y_{t+1} = p$. We can follow the same procedure in the case of the present problem too. Suppose that $y_t = p$. This implies that, since p is a constant, $y_{t+1} = p$ and $y_{t+2} = p$. Substituting these values into equation (9.3.1) we obtain $p = a_1 p + a_2 p + b$ or $p = b/(1-a_1-a_2)$, where $a_1 + a_2 \neq 1$. Therefore, following the derivation of equation (9.2.17), the particular solution in the present case can be written as

$$y_s = p = b/(1-a_1-a_2) \qquad (9.3.2)$$

If $a_1 + a_2 = 1$, then equation (9.3.2) becomes indeterminate and, in which case, we will have to use, as mentioned in Section 9.2.3, forms (instead of the form $y_t = p$) such as $y_t = pt$, $y_t = pt^2$, etc.

Having determined the particular solution to equation (9.3.1), we can now attempt to determine the complementary function of the homogenous form ($b = 0$) of the equation. The reader must have noticed that we obtained the solution to equation (9.2.1) as $y_t = y_c = Ca^t$ and called it the complementary function. Let us write, for convenience, the last equation as $y_t = y_c = Cv^t$, where $v = a$. We shall, therefore, assume that $y_t = Cv^t$ in the present case too, which implies that $y_{t+1} = Cv^{t+1}$ and $y_{t+2} = Cv^{t+2}$. Substituting these into the homogenous but alternative form of equation (9.3.1), $y_{t+2} - a_1 y_{t+1} - a_2 y_t = 0$, we obtain $Cv^{t+2} - a_1 Cv^{t+1} - a_2 Cv^t = 0$ or $Cv^t[v^2 - a_1 v - a_2] = 0$. Since Cv^t cannot be zero, then the last equation implies

$$v^2 - a_1 v - a_2 = 0. \tag{9.3.3}$$

Notice that equation (9.3.3) is a quadratic equation, similar to equation (1.6.8), which is called the characteristic equation of the difference equation $y_{t+2} - a_1 y_{t+1} - a_2 y_t = 0$. We can apply the quadratic formula in equation (1.6.9) to find the characteristic roots of equation (9.3.3). Since equation (9.3.3) is a quadratic equation, it possesses two roots. Applying quadratic formula we obtain the roots as

$$v_1, v_2 = \frac{-a_1 \pm \sqrt{a_1^2 - 4\times 1\times a_2}}{2\times 1} = \frac{-a_1 \pm \sqrt{a_1^2 - 4a_2}}{2} = \frac{-a_1}{2} \pm \frac{\sqrt{a_1^2 - 4a_2}}{2}. \tag{9.3.4}$$

An astute reader must have noticed that there may arise three different situations with regard to the roots depending upon the value of the square-root expression in equation (9.3.4). We may obtain real and distinct roots; real and repeated roots; or complex roots. Each of these is explained below.

Real and distinct roots

Suppose that $a_1^2 > 4a_2$. Then we will obtain two real and distinct values for the roots v_1 *and* v_2. In this event, expressions v_1^t and v_2^t (in $y_t = Cv_1^t$ and $y_t = Cv_2^t$, respectively) will be linearly independent. Therefore, the complementary function can be written as

$$y_c = C_1 v_1^t + C_2 v_2^t, \tag{9.3.5}$$

where C_1 *and* C_2 are constants. Therefore, we can now write the complete solution to equation (9.3.1) by combining the complementary function [equation (9.3.5)] and the particular solution [equation (9.3.2)] as

$$y_t = y_c + y_s = C_1 v_1^t + C_2 v_2^t + \frac{b}{1 - a_1 - a_2}. \tag{9.3.6}$$

Notice that we can determine the values of the constants C_1 *and* C_2 through initial conditions.

Real and repeated roots

Suppose that $a_1^2 = 4a_2$. Then equation (9.3.4) implies that $v_1, v_2 = -a_1/2$; that is, the two roots are real and repeated (or equal). Since $v_1 = v_2 = -a_1/2$, we can write $v = v_1 = v_2 = -a_1/2$. Then we may be tempted to write the complementary function, as we did in the case of equation (9.3.5), as $y_c = C_1 v_1^t + C_2 v_1^t = (C_1 + C_2)v_1^t = (C_1 + C_2)v^t = C_3 v^t$, where $C_3 = C_1 + C_2$. But, although we need two constants, this complementary function contains only one constant C_3. Therefore, this complementary function will not constitute a solution to the homogenous form of difference equation (9.3.1). One way out of this problem is to use an additional term of the form $C_4 t v^t$, where C_4 is a constant. Adding this term to $y_c = C_3 v^t$, we obtain the complementary function as

$$y_c = C_3 v^t + C_4 t v^t. \tag{9.3.7}$$

Therefore, we can now write the general or complete solution to equation (9.3.1) by combining the complementary function [equation (9.3.7)] and the particular solution [equation (9.3.2)] as

$$y_t = y_c + y_s = C_3 v^t + C_4 t v^t + \frac{b}{1 - a_1 - a_2}. \tag{9.3.8}$$

Notice that, as before, we can determine the values of the constants C_3 *and* C_4 through initial conditions.

Complex roots

As the last case, suppose that $a_1^2 < 4a_2$. In this event, the roots become complex numbers. We know from Section S1.12.6 that complex numbers have both real and imaginary parts. Notice that we can write equation (9.3.4) as $v_1, v_2 = [-a_1/2] \pm \{[\sqrt{(4a_2 - a_1^2) \times -1}]/2\} = [-a_1/2] \pm \{[\sqrt{(4a_2 - a_1^2)} \times \sqrt{-1}]/2\} = [-a_1/2] \pm \{[\sqrt{(4a_2 - a_1^2)} \times i]/2\}$, where $i = \sqrt{-1}$. Notice also that the equation has a real part $-a_1/2$ and an imaginary part $\sqrt{(4a_2 - a_1^2)} \times i/2$, making v_1 *and* v_2 complex numbers. Instead of denoting the opposite side of a triangle by v as we did in Section S1.12, we denote it here by q to distinguish it from the characteristic root notation v and, thereby, to avoid notational confusion. Let us now denote the real part of the complex number by $u = -a_1/2$ and the imaginary part by $qi = \sqrt{(4a_2 - a_1^2)} \times i/2$, where $q = \sqrt{(4a_2 - a_1^2)}/2$. Therefore, the two complex roots can be written as $v_1, v_2 = u \pm qi$. This implies that the complementary function of the solution to equation (9.3.1) can be written as

$$y_c = C_1 v_1^t + C_2 v_2^t = C_1(u + qi)^t + C_2(u - qi)^t, \tag{9.3.9}$$

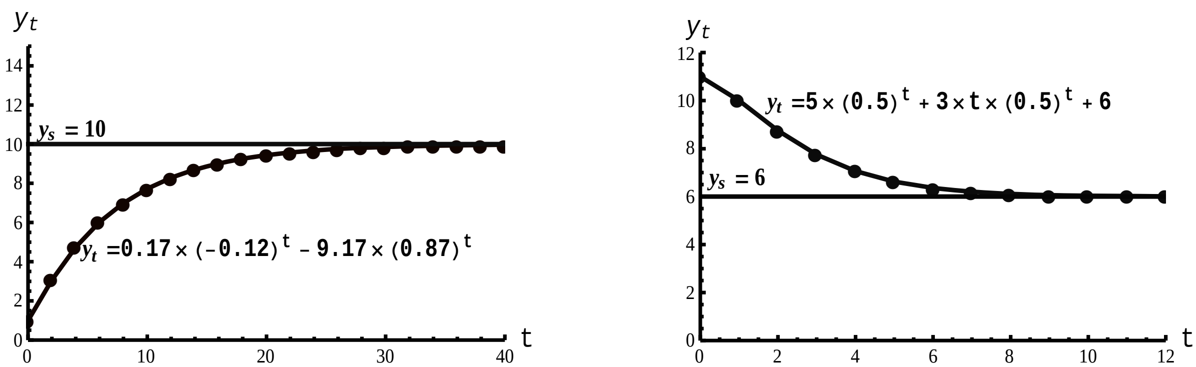
where C_1 *and* C_2 are constants. We found in Section S1.12.3 (using h, instead of r, for hypotenuse in Figure 1.12.4) that $\cos\theta = u/h$ and $\sin\theta = q/h$, which implied that $u = h\cos\theta$ and $q = h\sin\theta$. This implies that we can write $u + qi$ and $u - qi$ in equation (9.3.9) as $h\cos\theta + hi\sin\theta$ and $h\cos\theta - hi\sin\theta$, respectively. Therefore, equation (9.3.9) can be rewritten as $y_c = C_1(h\cos\theta + hi\sin\theta)^t + C_2(h\cos\theta - hi\sin\theta)^t$. We can now use De Moivre's theorem introduced in Section S1.12.6, to convert the last equation into the form $y_c = h^t C_1(\cos\theta t + i\sin\theta t) + h^t C_2(\cos\theta t - i\sin\theta t)$, which can be simplified further to obtain

$$y_c = h^t(C_3 \cos\theta t + C_4 \sin\theta t), \tag{9.3.10}$$

where $C_3 = C_1 + C_2$ and $C_4 = (C_1 - C_2)i$, and $h = \sqrt{u^2 + q^2}$. Having found the complementary function, we can now write the complete solution to equation (9.3.1) by combining the complementary function [equation (9.3.10)] and the particular solution [equation (9.3.2)] as

$$y_t = y_c + y_s = h^t(C_3\cos\theta t + C_4\sin\theta t) + \frac{b}{1 - a_1 - a_2}. \tag{9.3.11}$$

As an example, consider the difference equation $y_{t+2} = 0.75y_{t+1} + 0.1y_t + 1.5$, which is identical with equation (9.3.1) for $a_1 = 0.75, a_2 = 0.1,$ *and* $b = 1.5$. Since $1 - a_1 - a_2 = 0.15 \neq 1$, we can use equation (9.3.2) to obtain the particular solution: $y_s = p = b/(1 - a_1 - a_2) = 1.5/(1 - 0.75 - 0.1) = 10$. Let us now attempt to find the complementary function. Assume that we use the trial solution $y_t = Cv^t$, which implies that $y_{t+1} = Cv^{t+1}$ and $y_{t+2} = Cv^{t+2}$. With these forms we can write the homogenous form of the difference equation as $Cv^{t+2} - a_1Cv^{t+1} - a_2Cv^t = 0$, or as $Cv^t[v^2 - a_1v - a_2] = 0$. Then the characteristic equation can be written in the form of equation (9.3.3): $v^2 - a_1v - a_2 = 0$, or $v^2 - 0.75v - 0.1 = 0$. Since $a_1^2 = (0.75)^2 = 0.5625$ and $4a_2 = 4 \times 0.1 = 0.4$, $a_1^2 > 4a_2$ implying that the roots are both real and distinct. These roots can be found by applying equation (9.3.4): $v_1, v_2 = (-a_1 \pm \sqrt{a_1^2 - 4 \times a_2})/2 \times 1 = -0.12, 0.87$. Therefore, we can apply equation (9.3.5) to find the complementary function: $y_c = C_1v_1^t + C_2v_2^t = C_1(-0.12)^t + C_2(0.87)^t$. We can now combine the particular solution obtained earlier with this complementary function to obtain the complete solution in the form of equation (9.3.6): $y_t = y_c + y_s = C_1v_1^t + C_2v_2^t + b/(1 - a_1 - a_2) = C_1(-0.12)^t + C_2(0.87)^t + 10$. We shall now find the definite solution by definitizing the constants through initial conditions. Let us assume that $y_t = y_0 = 1$ when $t = 0$ and $y_t = y_1 = 2$ when $t = 1$. Then, from the solution $y_t = C_1(-0.12)^t + C_2(0.87)^t + 10$, we obtain $y_t = y_0 = C_1 + C_2 + 10 = 1$ and $y_t = y_1 = -0.12C_1 + 0.87C_2 + 10 = 2$, which can be solved simultaneously to yield $C_1 = 0.17$ *and* $C_2 = -9.17$. With these values we can rewrite the complete and definite solution as $y_t = 0.17(-0.12)^t - 9.17(0.87)^t + 10$ the graph of which is illustrated in Panel (A) of Figure 9.3.1.



(A) (B)

Figure 9.3.1

As another example, consider the difference equation $y_{t+2} = y_{t+1} - 0.25y_t + 1.5$, which is identical with equation (9.3.1) for $a_1 = 1, a_2 = -0.25,\ and\ b = 1.5$. Since $1 - a_1 - a_2 = 0.25 \neq 1$, we can use equation (9.3.2) to obtain the particular solution: $y_s = p = b/(1 - a_1 - a_2) = 1.5/(1 - 1 + 0.25) = 6$. Let us now attempt to find the complementary function. Assume that we use the trial solution $y_t = Cv^t$, which implies that $y_{t+1} = Cv^{t+1}$ and $y_{t+2} = Cv^{t+2}$. With these forms we can write the homogenous form of the difference equation as $Cv^{t+2} - a_1Cv^{t+1} + a_2Cv^t = 0$, or as $Cv^t[v^2 - a_1v + a_2] = 0$. Then the characteristic equation can be written in the form of equation (9.3.3): $v^2 - a_1v + a_2 = 0$, or $v^2 - v + 0.25 = 0$. Since $a_1^2 = 1^2 = 1$ and $4a_2 = 4 \times 0.25 = 1$, $a_1^2 = 4a_2$ implying that the roots are both real and repeated (equal). These roots can be found by applying equation (9.3.4): $v_1, v_2 = (-a_1 \pm \sqrt{a_1^2 - 4 \times a_2})/2 \times 1 = 0.5$. Therefore, we can apply equation (9.3.7) to find the complementary function: $y_c = C_1v_1^t + C_2tv_2^t = C_3(0.5)^t + C_4t(0.5)^t$. We can now combine the particular solution obtained earlier with this complementary function to obtain the complete solution in the form of equation (9.3.8): $y_t = y_c + y_s = C_3v_1^t + C_4tv_2^t + b/(1 - a_1 - a_2) = C_3(0.5)^t + C_4t(0.5)^t + 6$. We shall now find the definite solution by definitizing the constants through initial conditions. Let us assume that $y_t = y_0 = 11$ when $t = 0$ and $y_t = y_1 = 10$ when $t = 1$. Then, from the solution $y_t = C_3(0.5)^t + C_4t(0.5)^t + 6$, we obtain $y_t = y_0 = C_3 + 6 = 11$ and $y_t = y_1 = 0.5C_3 + 0.5C_4 + 6 = 10$, which can be solved simultaneously to yield $C_3 = 5\ and\ C_4 = 3$. With these values we can write the complete and definite solution as $y_t = 5(0.5)^t + 3t(0.5)^t + 6$ the graph of which is illustrated in Panel (B) of Figure 9.3.1.

As the last example, consider the difference equation $y_{t+2} = y_{t+1} - 0.5y_t + 1.5$, which is identical with equation (9.3.1) for $a_1 = 1, a_2 = -0.5,\ and\ b = 1.5$. Since $1 - a_1 - a_2 = 0.5 \neq 1$, we can use equation (9.3.2) to obtain the particular solution as $y_s = p = b/(1 - a_1 - a_2) = 1.5/(1 - 1 + 0.5) = 3$. Let us now attempt to find the complementary function. Assume that, as before, we use the trial solution $y_t = Cv^t$, which implies that $y_{t+1} = Cv^{t+1}$ and $y_{t+2} = Cv^{t+2}$. With these forms we can write the homogenous form of the difference equation as $Cv^{t+2} - a_1Cv^{t+1} + a_2Cv^t = 0$ or as $Cv^t[v^2 - a_1v + a_2] = 0$. Then the characteristic equation can be written as in the form of equation (9.3.3): $v^2 - a_1v + a_2 = 0$, or $v^2 - v + 0.5 = 0$. Since $a_1^2 = (-1)^2 = 1$ and $4a_2 = 4 \times 0.5 = 2$, $a_1^2 < 4a_2$ implying that the roots are complex numbers. Therefore, the two complex roots can be written as $v_1, v_2 = u \pm vi$, where $u = -a_1/2$ and $q = \sqrt{(4a_2 - a_1^2)}/2$. This implies that the complementary function can be written, following the procedure adopted in the derivation of equation (9.3.9), as $y_c = C_1v_1^t + C_2v_2^t = C_1(u + qi)^t + C_2(u - qi)^t$, which can be converted [as we derived equation (9.3.10)] into $y_c = h^t(C_3 \cos\theta t + C_4 \sin\theta t)$, where $C_3 = C_1 + C_2$ and $C_4 = (C_1 - C_2)i$, and $h = \sqrt{u^2 + q^2}$. Since $u^2 = a_1^2/4$ and $q^2 = (4a_2 - a_1^2)/4$, $h = \sqrt{u^2 + q^2} = \sqrt{(a_1^2 + 4a_2 - a_1^2)/4} = \sqrt{a_2} = \sqrt{0.5} = 0.707$. We know that $\cos\theta = u/h = 0.5/0.707 = 0.707$ and $\sin\theta = q/h = 0.5/0.707 = 0.707$, which gives us $\theta \cong 0.785$. Therefore, the complementary function can be rewritten as $y_c = (0.707)^t(C_3 \cos 0.785t + C_4 \sin 0.785t)$. We can now combine the particular solution obtained earlier with this complementary function to obtain the complete solution as $y_t = y_c + y_s = (0.707)^t(C_3 \cos 0.785t + C_4 \sin 0.785t) + 3$. Assume, for convenience, that $C_3 = 1$ *and* $C_4 = 1$. Then the complete and definite solution can be written as $y_t = (0.707)^t(\cos 0.785t + \sin 0.785t) + 3$, the graph of which is illustrated in Panel (A) of Figure 9.3.2. Notice that this graph is for continuous t. But, in difference equation we treat t to be discrete and, then, instead of a continuous graph, we will obtain a graph that corresponds to the graph of a step-function.

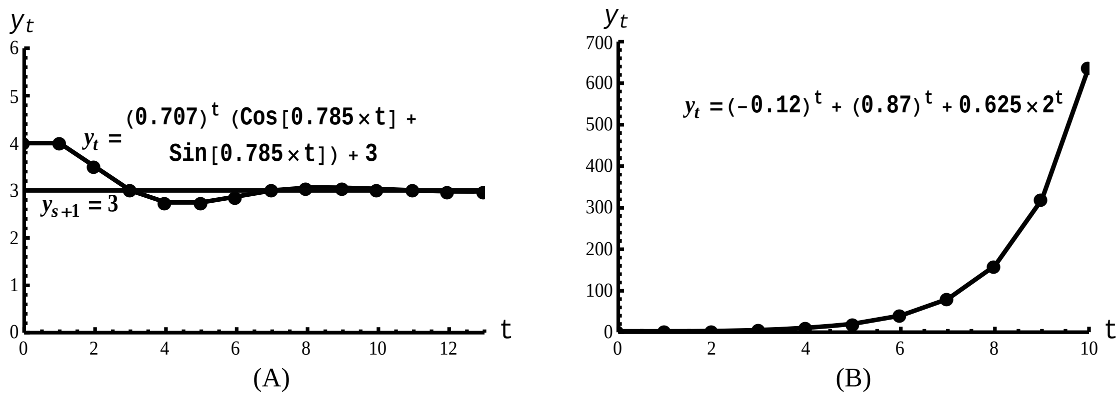


Figure 9.3.2

9.3.3 Convergence of second-order, linear, autonomous difference equations

We know from our exposition of first-order, linear, autonomous difference equations that their solutions consisted of two parts: the complementary function (y_c) and the particular solution (y_s). Therefore, we denoted the complete solution as $y_t = y_c + y_s$. We defined the term y_s as steady-state value, the value to which or from which y_t converged (if it ever converged) or diverged (if it diverged). This implies that we treated y_s as the reference point for convergence or divergence. Similarly, the solution to a general homogenous difference equation was denoted by $y_t = y_c$ because $y_s = 0$ in its solution (as $b = 0$). These results are equally valid in the case of second-order, linear, autonomous difference equations too.

One of our major points, as stated in Section 9.2.4, was that convergence or divergence of the path of y_t depended only on the absolute value of the coefficient, a, of the difference equation. We observed that if the absolute value of the coefficient a was greater than 1, then the time path would diverge; and if the absolute value of a was less than 1, then the time path would converge. This result is also valid in the case of a second-order, linear, autonomous difference equation. But, in the case of such equations, we have to take into account the absolute values of two roots (v_1 *and* v_2) and not the absolute value of just one (a). As we will see below, y_t will converge to (or diverge from) the steady-state value if the absolute values of the roots (irrespective of them being real and distinct, real and repeated, or complex) are less than 1. This steady-state value will be $y_s = b/(1 - a_1 - a_2)$ if $b \neq 0$ and will be zero (that is, $y_s = 0$ or converge to the horizontal axis) if $b = 0$.

We found in the last section that there would be two characteristic roots (v_1 *and* v_2) in the case of a second-order difference equation. We also found that these roots could be real and distinct roots, real and repeated roots, or complex roots. Let us first consider the case of real and distinct roots. In this case, $v_1 \neq v_2$. If the absolute values of both of these roots are greater than 1, then y_c part (or $C_1v_1^t + C_2v_2^t$) of equation (9.3.6) will increase as t increases. This implies that y_t will diverge from y_s (if $b \neq 0$) or from the horizontal axis (if $b = 0$). If the absolute values of both roots are less than one, then y_c part of equation (9.3.6) will decrease as t increases. This implies that y_t will converge to y_s (if $b \neq 0$) or to the horizontal axis (if $b = 0$). This is illustrated for $b \neq 0$ in Panel (A) of Figure 9.3.1. If $|v_1| < 1$ *and* $|v_2| > 1$, then the term $C_1v_1^t$ of y_c will decrease while the term $C_2v_2^t$ of y_c will increase (as t increases) causing y_c, and there by y_t, to deviate or diverge from the steady-state y_s.

We shall now consider the convergence of the solutions to second-order, linear, autonomous difference equations when the characteristic roots are real but repeated. The general solution with real and repeated roots is given in equation (9.3.8). If $|v|<1$, both $C_3 v^t$ *and* $C_4 t v^t$ terms of y_c will decrease as t increases (even if the second term, $C_4 t v^t$, is multiplied by t) and, therefore, y_t will converge to y_s (if $b \neq 0$) or to the horizontal axis (if $b = 0$). This is illustrated for $b \neq 0$ in Panel (B) of Figure 9.3.1. It is needless to state that y_t will diverge if $|v = v_1 = v_2| > 1$.

Lastly, we can consider the convergence of the solutions to second-order, linear, autonomous difference equations when the characteristic roots are complex numbers. For this we can use the solution in equation (9.3.11). Notice that the terms inside the brackets of this equation (which is equal to y_c) will give us fluctuating but step-like graphs. However, whether y_t will converge to or diverge from the steady-state value or not when $t \to \infty$ and $b \neq 0$ (or to horizontal axis, or to zero when $b = 0$) is still determined by the term h^t, where h is the absolute value of the complex roots. If $h < 1 (h > 1)$, then y_t will converge to (diverge from) y_s (if $b \neq 0$) and to zero if $b = 0$. This is illustrated for $b \neq 0$ in Panel (A) of Figure 9.3.2.

9.3.4 Second-order, linear, non-autonomous difference equations

So far in the present section we were concerned with the solutions (and their convergence or divergence) to second-order, linear, autonomous difference equations. Let us now consider the solutions to second-order, linear, non-autonomous difference equations with variable terms.

As a general case, consider the second-order, linear, autonomous difference equation (9.3.1): $y_{t+2} = a_1 y_{t+1} + a_2 y_t + b$. If we assume that the term b in the last equation is a function of time t (and, therefore, the equation is non-autonomous), then the equation becomes as a difference equation with a variable term. Suppose that the general form of such an equation is given by

$$y_{t+2} = a_1 y_{t+1} + a_2 y_t + b m^t, \tag{9.3.12}$$

where a_1, a_2, b *and* m are constants. Notice that we now have, instead of a constant b, a term that is a function of time in the difference equation. Since the coefficients are still constants, the move from a constant term to a variable term will only affect the particular solution. This implies that the complementary function still remains the same as in the case of difference equations with constant terms. This also implies that we need only attempt to determine the new particular solution. But, how do we do this?

In the case of equations with constant coefficients, we used the trial solution $y_t = p$, $y_{t+1} = p$, $y_{t+2} = p$, etc., to find the particular solution. Since the term is a variable in the present case, let us use the trial solution $y_t = Bm^t$, which implies that $y_{t+1} = Bm^{t+1}$ and $y_{t+2} = Bm^{t+2}$. Substituting the last three equations into equation (9.3.12), we obtain $Bm^{t+2} = a_1 Bm^{t+1} + a_2 Bm^t + bm^t$, which can be simplified to obtain $Bm^t = bm^t / (m^2 - a_1 m - a_2)$. Therefore, we can write the particular solution as

$$y_s = Bm^t = \frac{bm^t}{m^2 - a_1 m - a_2} \tag{9.3.13}$$

where it is assumed that the denominator is not equal to zero. Notice that the particular solution is a function of t. Therefore, it is referred to a **moving equilibrium**. If $m^2 - a_1 m - a_2 = 0$, then we will have to try other forms such as $y_t = Btm^t$, $y_t = Bt^2 m^t$, etc. Therefore, we can now write the complete solution to equation (9.3.12) by combining the complementary function [equation (9.3.5) if the roots are real and distinct; equation (9.3.7) if the roots are real and repeated; and equation (9.3.10) if the roots are complex numbers] and the particular solution in equation (9.3.13) as

$$y_t = y_c + y_s = y_c + \frac{bm^t}{m^2 - a_1 m - a_2}. \tag{9.3.14}$$

Suppose now, instead of equation (9.3.12), we have a second-order, linear, non-autonomous difference equation of the form

$$y_{t+2} = a_1 y_{t+1} + a_2 y_t + bt^n , \tag{9.3.15}$$

where a_1, a_2, b *and* n are positive integers. As we stated above, the complementary function will remain as before and the change will be only in the particular solution. Again as before, let us try the trial solution $y_t = B_0 + B_1 t + B_2 t^2 + ... + B_n t^n$. This implies that $y_{t+1} = B_0 + B_1(t+1) + B_2(t+1)^2 + \quad ... + B_n(t+1)^n$, $y_{t+2} = B_0 + B_1(t+2) + B_2(t+2)^2 + ... + B_n(t+2)^n$, …., and $y_{t+n} = B_0 + B_1(t+n) + B_2(t+n)^2 + \ ... + B_n(t+n)^n$. Now suppose, as a special case, that $n = 1$ in equation (9.3.15). Therefore, equation (9.3.15) becomes

$$y_{t+2} = a_1 y_{t+1} + a_2 y_t + bt \tag{9.3.16}$$

Then the trial solutions become $y_t = B_0 + B_1 t$, $y_{t+1} = B_0 + B_1(t+1)$, and $y_{t+2} = B_0 + B_1(t+2)$. Substituting these trial solutions into equation (9.3.16) we obtain $B_0 + B_1(t+2) = a_1[B_0 + B_1(t+1)] + a_2[B_0 + B_1 t] + bt$, or $B_0 + B_1(t+2) - a_1[B_0 + B_1(t+1)] - a_2[B_0 + B_1 t] = bt$. The last equation can be written as $(1 - a_1 - a_2)B_0 + [(t+2) - a_1(t+1)]B_1 - a_2 B_1 t = bt$. We can now apply the method of undetermined coefficients used in Section 8.3.9 to obtain $(1 - a_1 - a_2)B_0 + (2 - a_1)B_1 = 0$ and $(1 - a_1 - a_2)B_1 = b$, which can give us the values of B_0 *and* B_1 denoted by b_0 *and* b_1. With these values we can write the particular solution

$$y_s = b_0 + b_1 t . \tag{9.3.17}$$

Notice that if the trial solution $y_t = B_0 + B_1 t + B_2 t^2 + ... + B_n t^n$ does not work, we will have to resort to forms such as $y_t = t(B_0 + B_1 t + B_2 t^2 + ... + B_n t^n)$, $y_t = t^2(B_0 + B_1 t + B_2 t^2 + ... + B_n t^n)$, etc. Therefore, we can now write the complete solution to equation (9.3.16) by combining the complementary function [equation (9.3.5) if the roots are real and distinct; equation (9.3.7) if the roots are real and repeated; and equation (9.3.10) if the roots are complex numbers] and the particular solution in equation (9.3.17) as

$$y_t = y_c + y_s = y_c + b_0 + b_1 t . \tag{9.3.18}$$

As an example, consider the second-order, linear, non-autonomous difference equation $y_{t+2} = 0.75 y_{t+1} + 0.1 y_t + 1.5 \times 2^t$. This equation is identical, if $t = 0$ in the term, with the one we solved in the first example in Section 9.3.2. It is also identical with equation (9.3.12) for $a_1 = 0.75, a_2 = 0.1, b = 1.5$ *and* $m = 2$. We have obtained the complementary function of this equation in Section 9.3.2 as $y_c = C_1(-0.12)^t + C_2(0.87)^t$. We now need to find out the particular solution. For this we can apply equation (9.3.13) and obtain it as $y_s = 1.5m^t / (2^2 - 0.75 \times 2 - 0.1) = 0.625 \times 2^t$. Therefore, applying equation (9.3.14), we can write the complete solution as $y_t = y_c + y_s = C_1(-0.12)^t \ + C_2(0.87)^t + 0.625 \times 2^t$, whose graph is illustrated in Panel (B) Figure 9.3.2 for $C_1 = C_2 = 1$. Although the solution to the same difference equation with constant term converged to the steady-state [Panel (A) of Figure 9.3.1], the path of the solution diverges when we replace the constant term with a variable term [Panel (B) of Figure 9.3.2].

As another example, consider the second-order, linear, non-autonomous difference equation $y_{t+2} = 0.75 y_{t+1} + 0.1 y_t + 1.5t$, which is identical with the difference equation, with $t = 1$ in the term, in the first problem we solved in Section 9.3.2, and which is also identical with the difference equation (9.3.12) for $a_1 = 0.75, a_2 = 0.1, b = 1.5$ *and* $n = 1$. We have obtained the complementary function in Section 9.3.2: $y_c = C_1(-0.12)^t + C_2(0.87)^t$. We now need to find out the particular solution. For this we can apply equation (9.3.17). But, to apply this equation we need to first find out b_0 *and* b_1. These can be found by solving equations $(1 - a_1 - a_2)B_0 = 0$, $(2 - a_1)B_1 = 0$, and $(1 - a_1 - a_2)B_1 = b$. When we substitute the constants, we have $(1 - 0.75 - 0.1)B_0 + (2 - 0.75)B_1 = 0$ and $(1 - 0.75 - 0.1)B_1 = 1.5$. The last equation gives $B_1 = 10$. Substituting this value into $(1-0.75-0.1)B_0+(2-0.75)B_1=0$ and simplifying yield $B_0=-83.33$ and, thus $b_0=-83.33$ *and* $b_1=10$. Therefore, the particular solution in the present example can be written, using equation (9.3.17) as $y_s=b_0+b_1t=-83.33+10t$. Thus, applying equation (8.3.18) we can write the complete solution as $y_t=y_c+y_s=C_1(-0.12)^t+C_2(0.87)^t-83.33+10t$, the graph of which is illustrated in Panel (A) of Figure 9.3.3 for $C_1=C_2=1$. Compare this graph with the graph in Panel (A) of Figure 9.3.1.

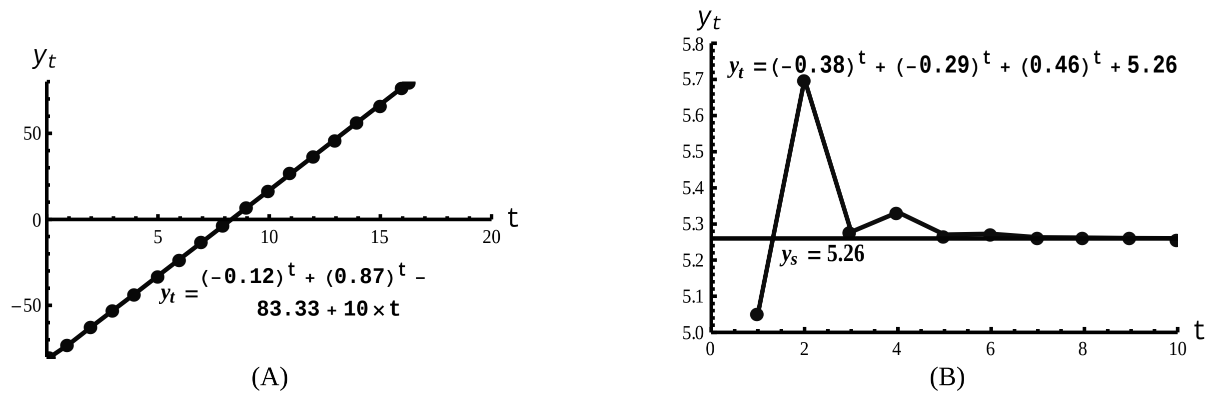


Figure 9.3.3

9.3.5 Higher-order, linear, autonomous difference equations: solutions

So far in the present chapter we were concerned with the solutions (and their convergence) to difference equations of order up to two. Let us now consider the solutions to linear, autonomous difference equations of order higher than two. Consider first the general n[th]-order, linear, autonomous difference equation $y_{t+n}=a_1y_{t+n-1}+a_2y_{t+n-2}+...+a_{n-1}y_{t+n-(n-1)}+a_ny_{t+n-n}+b$, or

$$y_{t+n}=a_1y_{t+n-1}+a_2y_{t+n-2}+...+a_{n-1}y_{t+1}+a_ny_t+b \tag{9.3.19}$$

As in the case of the general solution to the first-order (second-order), linear, autonomous difference equation presented in equation (9.2.18) [equation (9.3.6)], the solution to an n[th]-order, linear, autonomous difference equation, such as equation (9.3.19), also involves two parts: the complementary function and the particular solution. Let us first determine the complementary function of the solution to an n[th] -order equation such as equation (9.3.19).

Recall that we first attempted trial solutions to the first-and-second-order equations before we arrived at equations (9.2.18) and (9.3.6). We can follow the same trial solution in the case of the homogenous form of equation (9.3.19) too. Suppose that we try the solution $y_t=Cv^t$, which implies that $y_{t+1}=Cv^{t+1}$, $y_{t+n-2}=Cv^{t+n-2}$, …, $y_{t+n-1}=Cv^{t+n-1}$, and $y_{t+n}=Cv^{t+n}$. If we substitute these equations into the homogenous form of equation (9.3.19), we obtain $Cv^{t+n}=a_1Cv^{t+n-1}+a_2Cv^{t+n-2}+...+a_{n-1}Cv^{t+1}+a_nCv^t$ or $Cv^{t+n}-a_1Cv^{t+n-1}-a_2Cv^{t+n-2}-...-a_{n-1}Cv^{t+1}-a_nCv^t=0$; or $Cv^t[v^n-a_1v^{n-1}-a_2v^{n-2}-...-a_{n-1}v-a_n]=0$. Since Cv^t cannot be zero, then the last equation can be written as $v^n-a_1v^{n-1}-a_2v^{n-2}-...-a_{n-1}v-a_n=0$, which is an n[th]-degree polynomial and is the characteristic equation of the difference equation (9.3.19). If this characteristic equation is solved, we will obtain n characteristic roots $v_1,v_2,...,v_n$. As we found in Section 9.3.2, these roots may be real and distinct; or real and repeated; or complex. If the roots are real and distinct, we can write the complementary function, following equation (9.3.5), as

$$y_c=C_1v_1^t+C_2v_2^t+...+C_nv_n^t=\sum_{i=1}^{n}C_iv_i^t\ . \tag{9.3.20}$$

Notice that equation (9.3.20) is simply a generalization of equation (9.3.5). If the roots are real but repeated, we can write the complementary function, following equation (9.3.7), as

$$y_c = C_1 v_1^t + C_2 t v_1^t + ... + C_n t^{n-1} v_n^t = \sum_{i=1}^{n} C_i t^{i-1} v_i^t \,. \tag{9.3.21}$$

If any of the two roots, say v_{n-1} *and* v_n, happen to be complex, then we can apply equation (9.3.10) to obtain the complementary function corresponding to these complex roots as $y_{c,(n-1),n} = h^t (C_{n-1} \cos\theta t + C_n \sin\theta t)$, which can be combined with equation (9.3.20) to obtain

$$\begin{aligned} y_c &= C_1 v_1^t + C_2 v_2^t + ... + C_{n-2} v_{n-2}^t + h^t [C_{n-1} \cos\theta\, t + C_n \sin\theta\, t] \\ &= \sum_{i=1}^{n-2} C_i v_i^t + h^t [C_{n-1} \cos\theta t + C_n \sin\theta t]. \end{aligned} \tag{9.3.22}$$

Let us now attempt to determine the particular solution for which we can use equation (9.3.19). In the case of both first-order and second-order difference equations, we used trial solutions such as $y_t = p$ to find the respective particular solutions presented in equations (9.2.17) and (9.3.2). We can use the same procedure to find the particular solution to equation (9.3.19). Suppose that we use the trial solution $y_t = p$, which implies that $y_{t+1} = p$, …, $y_{t+n-2} = p$, $y_{t+n-1} = p$, and $y_{t+n} = p$. Substituting these values into equation (9.3.19) we obtain $p = a_1 p + a_2 p + ... + a_{n-1} p + a_n p + b$; or $p - a_1 p - a_2 p - ... - a_{n-1} p - a_n p = b$; or $p[1 - a_1 - a_2 - ... - a_{n-1} - a_n] = b$. Solving the last equation yields the particular solution

$$y_s = p = \frac{b}{1 - a_1 - a_2 - ... - a_{n-1} - a_n}, \tag{9.3.23}$$

where $[1 - a_1 - a_2 - ... - a_{n-1} - a_n] \neq 0$. If $[1 - a_1 - a_2 - ... - a_{n-1} - a_n] = 0$, as stated in Sections 9.2.3 and 9.3.2, we will have to use trial solutions such as $y_t = pt$, $y_t = pt^2$, etc.

Since we obtained both the complementary function and the particular solution, we can now combine them to write the complete solution to the nth-order difference equation (9.3.19) as

$$y_t = y_c + y_s = \sum_{i=1}^{n} C_i v_i^t + \frac{b}{1 - a_1 - a_2 - ... - a_{n-1} - a_n} \tag{9.3.24}$$

if the roots are real and distinct; or as

$$y_t = y_c + y_s = \sum_{i=1}^{n} C_i t^{i-1} v^t + \frac{b}{1 - a_1 - a_2 - ... - a_{n-1} - a_n} \tag{9.3.25}$$

if the roots are real and repeated; or as

$$y_t = y_c + y_s = \sum_{i=1}^{n-2} C_i v_i^t + h^t [C_{n-1} \cos\theta t + C_n \sin\theta t] + \frac{b}{1 - a_1 - a_2 - ... - a_{n-1} - a_n}, \tag{9.3.26}$$

if two of the roots are complex.

As an example, consider the third-order, linear, autonomous, non-homogenous difference equation $y_{t+3} = -0.2 y_{t+2} + 0.2 y_{t+1} + 0.05 y_t + 5$. The homogenous form of this equation is $y_{t+3} + 0.2 y_{t+2} - 0.2 y_{t+1} - 0.05 y_t = 0$. Let us first find the complementary function using the trial solution $y_t = Cv^t$. If we substitute the last equation into the homogenous form of the difference equation, we obtain $Cv^{t+3} + 0.2Cv^{t+2} - 0.2Cv^{t+1} - 0.05Cv^t = 0$, which can be written as $Cv^t [v^3 + 0.2v^2 - 0.2v - 0.05] = 0$. Since Cv^t cannot be zero, the last equation can be written as $v^3 + 0.2v^2 - 0.2v - 0.05 = 0$, which is the characteristic equation of the present difference equation. If we solve this equation, we obtain the characteristic roots

$v_1 = -0.38, v_2 = -0.29,$ *and* $v_3 = 0.46$. These roots are real and distinct and, therefore, we can apply equation (9.3.20), with $n = 3$, to write the complementary function as $y_c = C_1 v_1^t + C_2 v_2^t + C_3 v_3^t$ $= C_1(-0.38)^t + C_2(-0.29)^t + C_3(0.46)^t$. The particular solution can be found by using equation (9.3.23). Since $a_1 = -0.2, a_2 = 0.2, a_3 = 0.05,$ the particular solution becomes $y_s = p = 5/[1-(-0.2)-0.2-0.05]$ $= 5/0.95 = 5.26$. Therefore, using equation (9.3.24), we can write the complete solution as $y_t = y_c + y_s = C_1(-0.38)^t + C_2(-0.29)^t + C_3(0.46)^t + 5.26$. Assuming, for simplicity, $C_1 = C_2 = C_3 = 1$, the solution can be definitized as $y_t = (-0.38)^t + (-0.29)^t + (0.46)^t + 5.26$ and the graph of which is shown in Panel (B) of Figure 9.3.3.

9.3.6 Higher-order, linear, autonomous difference equations: convergence

We observed from the solutions to difference equations so far that the time path given by the solution to the concerned variable would converge or diverge depending upon the value (s) of the characteristic root(s) of the difference equation. If the absolute value(s) of the root(s) is (are) less than one, then the time path will converge to the steady-state if the constant term $b \neq 0$ or to zero or the horizontal axis if the constant term $b = 0$. If the absolute value(s) of the root(s) is (are) greater than one, then the time path will diverge from the steady-state. Therefore, if one knows the absolute value(s) of the characteristic root(s), one can infer whether the time path of the variable will converge or diverge.

Finding the characteristic roots is not a big problem if the **characteristic polynomial** or the characteristic equation is of degree 2 or 3; that is, in the case of second-order or third-order difference equations. But, the problem will become more and more cumbersome as the degree of the polynomial increases. The question, then, is how one can determine whether the time path of the variable implied by a higher-order difference equation converges or diverges. One does not need to find the actual roots and then determine converges or divergence. One can obtain the same conclusion qualitatively, without actually finding the roots, using a theorem called **Schur theorem**. Since the proof of this theorem is beyond the scope of this book, we simply state it as follows. Schur theorem states that the absolute values of the roots of an nth-degree polynomial equation such as

$$a_0 v^n + a_1 v^{n-1} + a_2 v^{n-2} + \dots + a_{n-1} v + a_n = 0 \tag{9.3.27}$$

will be less than 1 (and then the time path of the variable of the difference equation will converge) if and only if the following n determinants are all positive. Notice the special nature of these determinants. Suppose that we split each of the above determinants into four $n \times n$ determinants by drawing two straight lines horizontally and vertically such that they intersect at the middle of the determinants. Then the determinant at the south-east part will be the mirror image of the determinant at the north-west part and vice versa; and the determinant at the south-west part will be the mirror image of the determinant at the north-east part and vice versa. Moreover, the diagonal elements in all the above five determinants are the same a_0, which is equal to 1 in our difference equation. This shows that it is fairly easy to construct and evaluate these determinants.

$$\mathbf{E_1}=\begin{vmatrix} a_0 & a_n \\ a_n & a_0 \end{vmatrix}, \quad \mathbf{E_2}=\begin{vmatrix} a_0 & 0 & a_n & a_{n-1} \\ a_1 & a_0 & 0 & a_n \\ a_n & 0 & a_0 & a_1 \\ a_{n-1} & a_n & 0 & a_0 \end{vmatrix}, \quad \mathbf{E_3}=\begin{vmatrix} a_0 & 0 & 0 & a_n & a_{n-1} & a_{n-2} \\ a_1 & a_0 & 0 & 0 & a_n & a_{n-1} \\ a_2 & a_1 & a_0 & 0 & 0 & a_n \\ a_n & 0 & 0 & a_0 & a_1 & a_2 \\ a_{n-1} & a_n & 0 & 0 & a_0 & a_1 \\ a_{n-2} & a_{n-1} & a_n & 0 & 0 & a_0 \end{vmatrix}, \dots,$$

$$\mathbf{E_n}=\begin{vmatrix} a_0 & 0 & \dots & 0 & a_n & a_{n-1} & \dots & a_1 \\ a_1 & a_0 & \dots & 0 & 0 & a_n & \dots & a_2 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ a_{n-1} & a_{n-2} & \dots & a_0 & 0 & 0 & \dots & a_n \\ a_n & 0 & \dots & 0 & a_0 & a_1 & \dots & a_{n-1} \\ a_{n-1} & a_n & \dots & 0 & 0 & a_0 & \dots & a_{n-2} \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ a_1 & a_2 & \dots & a_n & 0 & 0 & \dots & a_0 \end{vmatrix}.$$

Let us now apply the convergence test using Schur theorem stated above. For this we shall use the third-order difference equation solved in the last section: $y_{t+3} = -0.2y_{t+2} + 0.2y_{t+1} + 0.05y_t + 5$, or $y_{t+3} + 0.2y_{t+2} - 0.2y_{t+1} - 0.05y_t = 5$. We found there that the characteristic polynomial or equation of this difference equation was $v^3 + 0.2v^2 - 0.2v - 0.05 = 0$, which is identical with equation (9.3.27) for treat $n = 3$, $a_0 = 1, a_1 = 0.2, a_2 = -0.2,$ *and* $a_3 = -0.05$. Since there are four coefficients ($a_0, a_1, a_2,$ *and* a_3) in the present difference equation, we need to use determinants up to $\mathbf{E_3}$ given above. Then, the determinants, with the above values for the coefficients, become

$$\mathbf{E_1} = \begin{vmatrix} 1 & -0.05 \\ -0.05 & 1 \end{vmatrix}, \mathbf{E_2} = \begin{vmatrix} 1 & 0 & -0.05 & -0.2 \\ 0.2 & 1 & 0 & -0.05 \\ -0.05 & 0 & 1 & 0.2 \\ -0.2 & -0.05 & 0 & 1 \end{vmatrix}, \text{ and } \mathbf{E_3} = \begin{vmatrix} 1 & 0 & 0 & -0.05 & -0.2 & 0.2 \\ 0.2 & 1 & 0 & 0 & -0.05 & -0.2 \\ -0.2 & 0.2 & 1 & 0 & 0 & -0.05 \\ -0.05 & 0 & 0 & 1 & 0.2 & -0.2 \\ -0.2 & -0.05 & 0 & 0 & 1 & 0.2 \\ 0.2 & -0.2 & -0.05 & 0 & 0 & 1 \end{vmatrix}.$$

These determinants can be evaluated to obtain $\mathbf{E_1}$ = 0.9975 > 0, $\mathbf{E_2}$ = 0.959 > 0, and $\mathbf{E_3}$ = 0.871 > 0. Since all the determinants are positive, as per Schur theorem, all the characteristic roots must be less than one in absolute value. Therefore, the path of the solution to the difference equation $y_{t+3} = -0.2y_{t+2} + 0.2y_{t+1} + 0.05y_t + 5$ must converge. In fact, this was the result we found when we plotted the graph of the solution $y_t = (-0.38)^t + (-0.29)^t + (0.46)^t + 5.26$ to the difference equation in Panel (B) of Figure 9.3.3.

9.3.7 Application examples

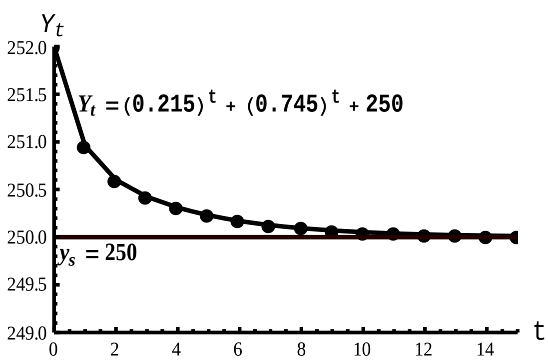
Example 1: Suppose that the aggregate income in an economy is given by $Y_t = C_t + I_t + G_t$, where Y_t, C_t, I_t *and* G_t denote aggregate income, consumption, investment and government expenditure, respectively, at time t. Also suppose that $C_t = \alpha Y_{t-1}; I_t = \beta(C_t - C_{t-1}) = \beta C_t - \beta C_{t-1}$ and that $G_t = G_0$, where α, β *and* G_0 denote constant marginal propensity to consume, constant **acceleration coefficient** or **accelerator**, and constant **autonomous government expenditure**, respectively. Set up the difference equation in income and solve it if $\alpha = 0.8, \beta = 0.2$ *and* $G_0 = 50$. Does national income converge or diverge? All the values are in billions of dollars.

Solution: This is an example of **Samuelson's multiplier-accelerator interaction model**. Let us first set up the difference equation. Substituting $C_t = \alpha Y_{t-1}, I_t = \beta(C_t - C_{t-1}) = \beta C_t - \beta C_{t-1},$ *and* $G_t = G_0$ into $Y_t = C_t + I_t + G_t$ we obtain $Y_t = \alpha Y_{t-1} + \beta C_t - \beta C_{t-1} + G_0$. Since $C_t = \alpha Y_{t-1}$ and $C_{t-1} = \alpha Y_{t-2}$, the last equation can be written as $Y_t = \alpha Y_{t-1} + \alpha\beta Y_{t-1} - \alpha\beta Y_{t-2} + G_0$ or as $Y_t = (\alpha + \alpha\beta)Y_{t-1} - \alpha\beta Y_{t-2} + G_0$. Substituting $\alpha = 0.8$ *and* $\beta = 0.2$, the last equation becomes $Y_t = 0.96Y_{t-1} - 0.16Y_{t-2} + 50$. Notice that this equation is identical with equation (9.3.1) with $Y = y, a_1 = 0.96, a_2 = -0.16,$ *and* $b = 50$. Moving time by two periods forward, the required difference equation can be written as $Y_{t+2} = 0.96Y_{t+1} - 0.16Y_t + 50$.

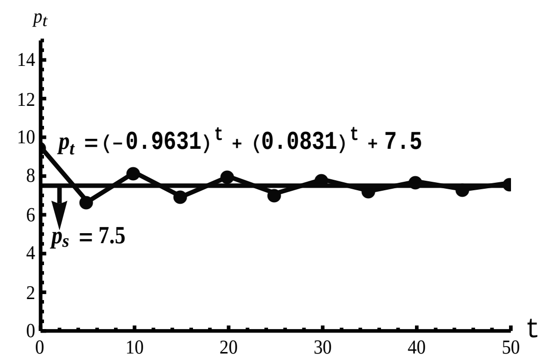
We can now find out the particular solution. For this we can use the trial solution that $Y_t = p$, which implies that $Y_{t+1} = p$ *and* $Y_{t+2} = p$. We shall now follow the same procedure as that we used to derive equation (9.3.2). Therefore, with the above values, we obtain the particular solution $Y_s = p = b/(1-a_1-a_2) = 50/[1-0.96-(-0.16)] = 250$. Therefore, the long-run (or the steady-state or equilibrium) level of national income in the economy is $250 billion.

Let us now find out the complementary function. For this, we use the trial solution $Y_t = Cv^t$, where C is a constant (and should not be confused with aggregate consumption), which implies that $Y_{t+1} = Cv^{t+1}$ *and* $Y_{t+2} = Cv^{t+2}$. Substituting these equations into the homogenous version of the difference equation $Y_{t+2} = 0.96Y_{t+1} - 0.16Y_t$ yields $Cv^{t+2} = 0.96Cv^{t+1} - 0.16Cv^t$, or $Cv^{t+2} - 0.96Cv^{t+1} + 0.16Cv^t = 0$, which can also be written as $Cv^t[v^2 - 0.96v + 0.16] = 0$. Therefore, we have $v^2 - 0.96v + 0.16 = 0$, which is the characteristic equation of the difference equation. Notice that the equation $v^2 - 0.96v + 0.16 = 0$ is identical with equation (9.3.3) with $a_1 = -0.96$ *and* $a_2 = 0.16$. Therefore, applying equation (9.3.4), we obtain the roots $v_1, v_2 = (-a_1/2) \pm (\sqrt{a_1^2 - 4a_2}/2) = 0.215, 0.745$, which are real and distinct. Therefore, we can write the complementary function, using equation (9.3.5), as $y_c = C_1v_1^t + C_2v_2^t = C_1(0.215)^t + C_2(0.745)^t$. We shall now combine the particular solution with the complementary function to obtain the complete solution $Y_t = Y_c + Y_s = C_1v_1^t + C_2v_2^t + b/(1-a_1-a_2) = C_1(0.215)^t + C_2(0.745)^t + 250$.

Notice that in our solution the absolute values of both roots are less than one ($v_1, v_2 = 0.215, 0.745$). Therefore, as per the results stated in Sections 9.3.3 and 9.3.6, the national income of the economy must converge to its steady-state value of $250 billion in the long-run. This is illustrated through the graph of the solution $Y_t = C_1(0.215)^t + C_2(0.745)^t + 250$, assuming $C_1 = C_2 = 1$, in Panel (A) of Figure 9.3.4.



(A)



(B)

Figure 9.3.4

Example 2: Suppose that the quantities supplied and demanded of a good at time t are given by $q_t^s = \alpha + \beta[p_{t-1} - \delta(p_{t-1} - p_{t-2})]$ and $q_t^d = \eta + \lambda p_t$, respectively, where p denotes price, and $\alpha, \beta,$ *and* δ denote constants. Find the difference equation for price assuming that the market clears and solve it assuming that $\alpha = 5, \beta = 8, \delta = -0.1, \eta = 20,$ *and* $\lambda = -10$. Does price converge to its steady-state value? All the prices are in dollars.

Solution: This is an example of a **cobweb model with price expectations**. Let us first set up the difference equation. Equating the demand and supply functions we obtain $\eta + \lambda p_t = \alpha + \beta[p_{t-1} - \delta(p_{t-1} - p_{t-2})]$. Substituting $\alpha = 5, \beta = 8, \delta = -0.1, \eta = 20,$ *and* $\lambda = -10$, simplifying, and moving time by two periods forward yield $p_{t+2} = -0.88p_{t+1} + 0.08p_t + 1.5$. Notice that the last equation is identical with equation (9.3.1) with $p = y, a_1 = 0.88, a_2 = -0.08,$ *and* $b = 1.5$.

We can now find out the particular solution. For this we use the trial solution that $p_t = p$, where p is a constant, which implies that $p_{t+1} = p$ *and* $p_{t+2} = p$. We now follow the same procedure as that we used to derive equation (9.3.2) to obtain the particular solution $p_s = p = 1.5/(1-a_1-a_2) = 1.5/[1-0.88-(-0.08)] = 7.5$. Therefore, the long-run (or the steady-state or equilibrium) price is $7.5.

Let us now find out the complementary function. For this, we shall use the trial solution $p_t = Cv^t$, where C is a constant, which implies that $p_{t+1} = Cv^{t+1}$ *and* $p_{t+2} = Cv^{t+2}$. Substituting these equations into the homogenous version of the difference equation yields $Cv^{t+2} = 0.88Cv^{t+1} - 0.08Cv^t$, or $Cv^{t+2} - 0.88Cv^{t+1} + 0.08Cv^t = 0$, which can also be written as $Cv^t[v^2 - 0.88v + 0.08] = 0$. From this we obtain $v^2 - 0.88v + 0.08 = 0$, which is the characteristic equation of the difference equation. Notice that this equation is identical with equation (9.3.3) with $a_1 = -0.88$ *and* $a_2 = 0.08$. Therefore, applying equation (9.3.4), we obtain the characteristic roots as $v_1, v_2 = (-a_1/2) \pm (\sqrt{a_1^2 - 4a_2})/2 = -0.9631, 0.0831$, which are real and distinct. Therefore, we can write the complementary function, using equation (9.3.5), as $p_c = C_1 v_1^t + C_2 v_2^t$, or $p_c = C_1(-0.9631)^t + C_2(0.0831)^t$. We can now combine the particular solution with the complementary function, using equation (9.3.6), to obtain the complete solution $p_t = p_c + p_s = C_1 v_1^t + C_2 v_2^t + b/(1 - a_1 - a_2) = C_1(-0.9631)^t + C_2(0.0831)^t + 7.5$.

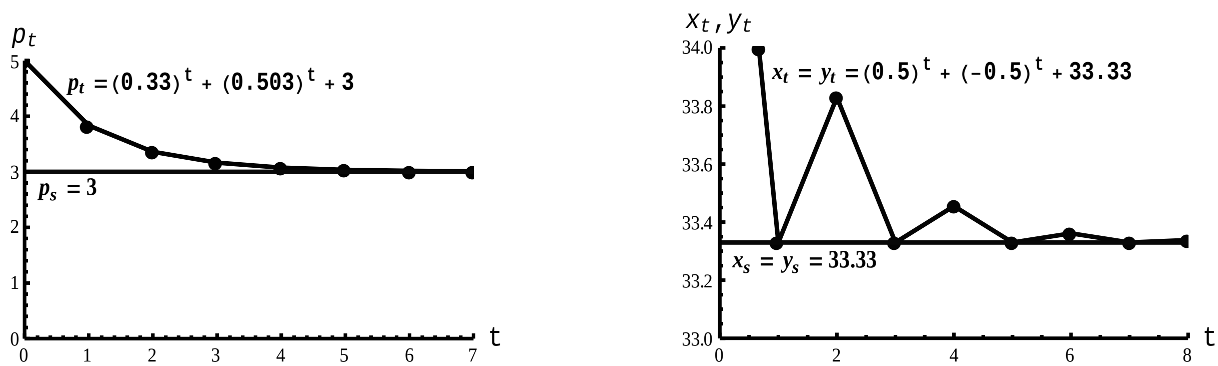
Notice that in our solution the absolute values of both roots are less than one ($v_1, v_2 = -0.9631, 0.0831$). Therefore, as per the results stated in Sections 9.3.3 and 9.3.6, the price of the good must converge to its steady-state value of $7.5 in the long-run. This is illustrated in the graph of the solution $p_t = C_1(-0.9631)^t + C_2(0.0831)^t + 7.5$, assuming $C_1 = C_2 = 1$, in Panel (B) of Figure 9.3.4.

Example 3: Assume that the relationship between the rate of growth of money wages at time $t\,(w_t)$ and the rate of unemployment at time $t\,(u_t)$ is given by $w_t = f(u_t) = \alpha - \beta u_t$, which is popularly called the **Phillips curve**. In the last function α *and* β are assumed to be constants and $\alpha > 0$ *and* $\beta > 0$. Phillips curve has been modified as a relationship between the rate of inflation at time $t\,(p_t)$ and the rate of unemployment at time $t\,(u_t)$ and is given by $p_t = f(u_t) - r$, where r denotes the constant productivity of labor. Combining the last two equations we obtain $p_t = f(u_t) - r = \alpha - \beta u_t - r$. This form has again been modified to include price expectations as $p_t = f(u_t) - r + p_t^e = \alpha - \beta u_t - r + p_t^e$, where p_t^e denotes **expected inflation** at time t. The last function represents the **expectations-augmented Phillips curve**. Also assume that **adaptive expectations** and **monetary policy** take the forms $p_{t+1}^e - p_t^e = h(p_t - p_t^e)$ and $u_{t+1} - u_t = -j(m - p_{t+1})$, respectively, where $h,\ j,$ *and* m are constants, and $h > 0$ *and* $j > 0$. Set up the difference equation in the rate of inflation and solve it if $\alpha = 10, \beta = 0.5, r = 10, h = 0.4, j = 10,$ *and* $m = 4$. Does p_t converge to, or diverge from, its steady-state (if it exists)?

Solution: Let us first substitute the values $\alpha = 10, \beta = 0.5, r = 10, h = 0.4, j = 10,$ *and* $m = 4$ into $p_t = \alpha - \beta u_t - r + p_t^e$, and $u_{t+1} - u_t = -j(m - p_{t+1})$ to obtain $p_t = p_t^e - 0.5u_t$, $p_{t+1}^e - p_t^e = 0.4(p_t - p_t^e) = 0.4p_t - 0.4p_t^e$, and $u_{t+1} - u_t = -10(4 - p_{t+1}) = -40 + 10p_{t+1}$, respectively. Notice that equation $p_t = p_t^e - 0.5u_t$ can also be written as $p_{t+1} = p_{t+1}^e - 0.5u_{t+1}$ and the former can be subtracted from the latter to obtain $p_{t+1} - p_t = (p_{t+1}^e - p_t^e) - 0.5(u_{t+1} - u_t)$. Substituting into this equations $p_{t+1}^e - p_t^e = 0.4p_t - 0.4p_t^e$ and $u_{t+1} - u_t = -40 + 10p_{t+1}$ yields $p_{t+1} - p_t = 0.4p_t - 0.4p_t^e - 5p_{t+1} + 20$ or $6p_{t+1} = 1.4p_t - 0.4p_t^e + 20$. Since $p_t^e = p_t + 0.5u_t$, the last equation can be written as $6p_{t+1} = p_t - 0.2u_t + 20$, which can be first-differenced to get $6[p_{t+1} - p_t] = [p_t - p_{t-1}] - 0.2[u_t - u_{t-1}]$. Since we can obtain $u_t - u_{t-1} = -10(4 - p_t)$ from $u_{t+1} - u_t = -10(4 - p_{t+1})$, we can substitute it into the last result to get $6[p_{t+1} - p_t] = [p_t - p_{t-1}] - 0.2[-40 + 10p_t]$; or $p_{t+1} = 0.833p_t - 0.166p_{t-1} + 1.33$; or $p_{t+2} = 0.833p_{t+1} - 0.166p_t + 1.33$, which is identical with equation (9.3.1) for $p = y, a_1 = 0.833, a_2 = -0.166,$ *and* $b = 1.33$. Therefore, the required difference equation is $p_{t+2} = 0.833p_{t+1} - 0.166p_t + 1.33$.

Let us now determine the particular solution. For this we shall use the trial solution $p_t = p$ (where p is a constant and should not be confused with p_t) implying that $p_{t+1} = p$ and $p_{t+2} = p$. If we substitute these trial solutions into the difference equation and simplify, we obtain an equation similar to the one from which we derived equation (9.3.2). Now using equation (9.3.2) we obtain the particular solution $p_s = b/(1-a_1-a_2) = 1.33/[1-0.833-(-0.166)] = 3$.

We can now determine the complementary function. For this we shall use the trial solution $p_t = Cv^t$, where C is a constant, implying $p_{t+1} = Cv^{t+1}$ and $p_{t+2} = Cv^{t+2}$. If we substitute these equations into the homogenous form of the difference equation, we obtain $Cv^{t+2} = 0.833Cv^{t+1} - 0.166Cv^t$; or $Cv^{t+2} - 0.833Cv^{t+1} + 0.166Cv^t = 0$; or $Cv^t[v^2 - 0.833v + 0.166] = 0$. Therefore, we have $v^2 - 0.833v + 0.166 = 0$, which is the characteristic equation of the difference equation, which can be solved to obtain the roots $v_1, v_2 = 0.33, 0.503$, which are real and distinct. Then, following equation (9.3.5), we can write the complementary function $p_c = C_1v_1^t + C_2v_2^t = C_1(0.33)^t + C_2(0.503)^t$. Following equation (9.3.6), we can now combine the particular solution with the complementary function to obtain the complete solution as $p_t = p_c + p_s = C_1(0.33)^t + C_2(0.503)^t + 3$. Since the absolute values of both characteristic roots are less than one, p_t will converge to its steady-state value of 3. This is illustrated, for $C_1 = C_2 = 1$, in Panel (A) of Figure 9.3.5.



(A) (B)

Figure 9.3.5

Example 4: Suppose that the two firms, x *and* y, in a **dynamic Cournot duopoly market** produce outputs x_t *and* y_t respectively at time t. The inverse demand function of each **duopolist** is given by $p_t = f(q_t) = 110 - q_t = 110 - x_t - y_t$, where $q_t = x_t + y_t$, and their average and marginal costs are the same and equal to \$10. We maintain the Cournot assumption that each firm chooses its output for period $t+1$, assuming that the other will continue with the quantity produced in period t, to maximize its profit. Set up the difference equations that represent the firm's reaction functions and solve them. Does the path of the solution converge to (or diverge from) the steady-state if it exists?

Solution: We are given that $p_t = f(q_t) = 110 - q_t = 110 - x_t - y_t$, which implies that the total revenue of firm x in period $t+1$ will be $R_{x,\,t+1} = 110x_{t+1} - x_{t+1}^2 - x_{t+1}y_t$, where R denotes total revenue; the total cost will be $C_{x,\,t+1} = 10x_{t+1}$, where C denotes total cost; and the total profit will be $\Pi_{x,\,t+1} = R_{x,\,t+1} - C_{x,\,t+1} = 100x_{t+1} - x_{t+1}^2 - x_{t+1}y_t$, where Π denotes total profit. Then the FOC for a maximum of profit for firm x in period $t+1$ is $\partial\Pi_{x,\,t+1}/\partial x_{t+1} = 100 - 2x_{t+1} - y_t = 0$, which can be simplified to obtain $x_{t+1} = 50 - 0.5y_t$ or $x_{t+2} = 50 - 0.5y_{t+1}$. The last equation represents the reaction function for firm x.

The total revenue of firm y in period $t+1$ will be $R_{y,\,t+1} = 110y_{t+1} - y_{t+1}^2 - x_ty_{t+1}$; the total cost will be $C_{y,\,t+1} = 10y_{t+1}$; and the total profit will be $\Pi_{y,\,t+1} = R_{y,\,t+1} - C_{x,\,t+1} = 100y_{t+1} - y_{t+1}^2 - y_{t+1}x_t$. Then the FOC for a maximum of profit for firm y in period $t+1$ is $\partial\Pi_{y,t+1}/\partial y_{t+1} = 100 - 2y_{t+1} - x_t = 0$, which can be simplified to obtain $y_{t+1} = 50 - 0.5x_t$ or $y_{t+2} = 50 - 0.5x_{t+1}$. The last equation represents the reaction function for firm y.

If we substitute the reaction function for firm y into that for firm x and vice versa, we obtain after simplifications $x_{t+2} = 0 \times x_{t+1} + 0.25x_t + 25$ and $y_{t+2} = 0 \times y_{t+1} + 0.25y_t + 25$. The last two equations are identical with equation (9.3.1) with $y = x, a_1 = 0, a_2 = 0.25,\ and\ b = 25$. Therefore, the required difference equations are $x_{t+2} = 0 \times x_{t+1} + 0.25x_t + 25$ and $y_{t+2} = 0 \times y_{t+1} + 0.25y_t + 25$.

To solve the difference equations, let us first determine the particular solution. Using the trial solution $x_t = p$, which implies $x_{t+1} = p$ and $x_{t+2} = p$ and substituting them into the difference equation we obtain, after using equation (9.3.2), the particular solution to firm $x's$ difference equation: $x_s = p = b/(1 - a_1 - a_2) = 25/(1 - 0 - 0.25) = 33.33$. By following the same procedure, we get the particular solution to firm $y's$ difference equation $y_s = p = b/(1 - a_1 - a_2) = 25/(1 - 0 - 0.25)$ = 3.33.

We can now determine the complementary functions. For this, we shall use the trial solution $x_t = Cv^t$, which implies that $x_{t+1} = Cv^{t+1}$ and $x_{t+2} = Cv^{t+2}$. Substituting these solutions into the homogeneous version of the difference equation yields $Cv^{t+2} = 0 \times Cv^{t+1} + 0.25Cv^t$ or $Cv^t[v^2 - 0v - 0.25] = 0$. Therefore, we have $v^2 - 0.25 = 0$, which is the characteristic equation of the difference equation. Solving $v^2 - 0.25 = 0$ yields $v_1, v_2 = 0.5, -0.5$, which are real and distinct. Therefore, applying equation (9.3.5), we can write the complementary function as $x_c = C_1v_1^t + C_2v_2^t = C_1(0.5)^t + C_2(-0.5)^t$. Let us now combine the particular solution with the complementary function, using equation (9.3.6), to obtain the complete solution: $x_t = C_1(0.5)^t + C_2(-0.5)^t + 33.33$. By following the same procedure, we can get $y_t = C_1(0.5)^t + C_2(-0.5)^t + 33.33$. Notice that the absolute values of the roots are less than one implying that both $x_t\ and\ y_t$ will converge to their common steady-state value of 33.33 as illustrated (assuming $C_1 = C_2 = 1$) in Panel (B) of Figure 9.3.5. Therefore, in the present example of the dynamic Cournot duopoly market, both firms produce equal quantities of 33.33 units in the long-run.

9.3.8 Exercises

1. Solve the following difference equations. Does y_t converge to or diverge from its steady-state, if it exists. Assume, for convenience, that all the constants in the solutions are equal to one.

(i) $y_{t+2} = 0.5y_{t+1} - 0.5y_t + 10$ (ii) $y_{t+2} = -0.5y_{t+1} + 10$ (iii) $y_{t+2} = 1.5y_t + 10$
(iv) $y_{t+2} = 2y_{t+1} - y_t + 10$ (v) $y_{t+2} = 2y_{t+1} + y_t + 10$ (vi) $y_{t+2} = 2y_{t+1} - 3y_t + 10$.

2. Solve the following difference equations.

(i) $y_{t+2} = 0.5y_{t+1} - 0.5y_t + 4^t$ (ii) $y_{t+2} = -0.5y_t + 2^t$
(iii) $y_{t+2} = 0.5y_{t+1} - 0.5y_t + t^2$ (iv) $y_{t+2} = 0.5y_{t+1} - 0.5y_t + 3 \times 4^t$.

3. Check whether y_t implied by the following difference equations converge or diverge.
(i) $y_{t+3} = 0.5y_{t+2} - 0.5y_{t+1} + y_t + 4$ (ii) $y_{t+3} = y_t + 4$
(iii) $y_{t+3} = -0.5y_{t+1} + y_t + 4$ (iv) $y_{t+3} = 0.5y_{t+2} + y_t + 4$.

4. **Application exercise**. Consider Samuelson's multiplier-accelerator interaction model solved in example 1 in Section 9.3.7. Suppose that the parameters are such that $\alpha = 0.5, \beta = 0.1\ and\ G_0 = \$30$. Set up the difference equation

in income and solve it assuming that the constants in the solution are all equal to 1. Does national income converge or diverge? All the values are in billions of dollars.

5. **Application exercise**. Consider the cobweb model with price expectations solved in example 2 in Section 9.3.7. Suppose, in the model, that $\alpha = 10, \beta = 5, \delta = -1/2, \eta = 10,$ *and* $\lambda = -8$. Does price converge to its steady-state value if it exists? All the prices are in dollars. Assume that all the constants in model equal 1.

6. **Application exercise**. Consider the model of expectations-augmented Phillips curve with adaptive expectations and monetary policy solved in example 3 in Section 9.3.7. Assume that, in the model, $\alpha = 12, \beta = 0.8, r = 8, h = 0.5, j = 5,$ *and* $m = 5$. Set up the difference equation in unemployment and solve it assuming that all the constants in the solution are equal to 1. Does unemployment rate converge to or diverge from its steady-state value, if it exists?

7. **Application exercise**. Consider the dynamic Cournot duopoly model solved in example 4 in Section 9.3.7. Suppose that the inverse demand function is $p_t = f(q_t) = 60 - q_t = 110 - x_t - y_t$, where $q_t = x_t + y_t$, and that the costs are zero. Set up the difference equations that represent the firms' reaction functions and solve them assuming that all the constants in the solution are equal to 1. Do the quantities produced by the firms converge to or diverge from equal or unequal steady-state values, if they exist?

8. **Application exercise**. Suppose that the quantities demand and supplied of a good are given by $q_{dt} = 200 - 4p_t$ and $q_{st} = 5p_t - 10 + 2(p_{t-1} - p_{t-2})$, where p_t denotes price at time t. Set up the difference equation in price and solve it assuming that all the constants in the solution are equal to 1. Explain the long-run behavior of price.

9.3.9 Mathematica applications

Higher-order difference equations can be solved in Mathematica using the command **RSolve**, used to solve first-order difference equations and introduced in Section 9.2 above. As demonstrations, consider the difference equations we solved in Section 9.3.2: $y_{t+2} = 0.75y_{t+1} + 0.1y_t + 1.5$, $y_{t+2} = y_{t+1} - 0.25y_t + 1.5$ and $y_{t+2} = y_{t+1} - 0.5y_t + 1.5$. Using the command **RSolve**, we can solve these difference equations without and with initial conditions to obtain respective results similar to those we obtained in Section 9.3.2 as

```
RSolve[y[t+2]==0.75×y[t+1]+0.1y[t]+1.5,y[t],t]
RSolve[y[t+2]==y[t+1]-0.25y[t]+1.5,y[t],t]
RSolve[y[t+2]==y[t+1]-0.5y[t]+1.5,y[t],t]
{{y[t]→10. +(-0.115535)^t C[1]+0.865535^t C[2]}}
{{y[t]→6. +0.5^t C[1]+0.5^t t C[2]}}
{{y[t]→3. +(0.5 -0.5 i)^t C[1]+(0.5 +0.5 i)^t C[2]}}


RSolve[{y[t+2]==0.75×y[t+1]+0.1y[t]+1.5,y[0]==1,y[1]==2},y[t],t]
RSolve[{y[t+2]==y[t+1]-0.25y[t]+1.5,y[0]==11,y[1]==10},y[t],t]
RSolve[{y[t+2]==y[t+1]-0.5y[t]+1.5,y[0]==1,y[1]==1},y[t],t]
{{y[t]→10. +0.214237 (-0.115535)^t-9.21424 0.865535^t}}
{{y[t]→6. +5. 0.5^t+3. 0.5^t t}}
{{y[t]→3. -(1. +1. i) (0.5 -0.5 i)^t-(1. -1. i) (0.5 +0.5 i)^t}}
```

9.4 Simultaneous Systems of Difference Equations

9.4.1 Introduction

So far in the present chapter we were concerned with the solutions to difference equations and the intertemporal behavior of these solutions. However, in all these analyses we were concerned with only one difference equation at a time. Our aim from now onwards in the present chapter is to solve, and determine the intertemporal behavior of the solutions of, difference equations by taking more than one equation at a time or by taking **simultaneous systems of difference equations** (hereafter, **SSDEs**).

One may wonder why we need to consider SSDEs. Notice that in most of the difference equations we considered so far we assumed that the value of a variable at a particular time depended on the value of the *same* variable at some other time. Consider, for example, the equation $y_{t+2} = a_1 y_{t+1} + a_2 y_t + b$, which can be written in functional form as $y_{t+2} = f(y_{t+1}, y_t)$. This function shows that the value of the variable y in period $t+2$ is a function of the value of the same variable in the past two periods and no other variable is involved in the function.

But, the fact is that the values of many variables in a particular time period may depend not only on the past values of the same variables but also on the values of some other variable(s) in the same or some other period(s) of time. Then we may get functions such as $y_{t+2} = g(y_{t+1}, y_t, x_t)$, $y_{t+2} = h(y_{t+1}, y_t, x_t, x_{t+1})$, $x_{t+2} = j(x_{t+1}, x_t, y_t)$, $x_{t+2} = k(x_{t+1}, x_t, y_{t+1}, y_t)$, etc. One can cite plenty of examples of this type of interdependence among variables over time from the fields of our interest. As an example, consider the inflation, unemployment and monetary policy model solved in example 3 in Section 9.3.7. In this model we observed that the expected inflation rate in a particular time period depended on both the unemployment and the expected inflation rates in the immediate past time period, and the unemployment rate in a particular time period, in turn, depended on the expected inflation and unemployment rates in the immediate past time period. Similarly, we observed in the dynamic Cournot duopoly model in example 4 of Section 9.3.7 that the quantity of a good produced by one of the duopolists in a particular time period depended on the quantity of output produced in the immediate past time period by the other duopolist. These examples imply that the difference equations involved are simultaneous in nature; or, in other words, we have SSDEs.

The question arising now is how one can solve such SSDEs and find out the time paths of the involved variables. As we will see shortly, the techniques of solving SSDEs are quite similar to those we used in solving single difference equations so far in the present chapter. There exist different methods to solve SSDEs. We will expose only two methods here: the **substitution method** and the **matrix method**.

9.4.2 Solution of SSDEs: the substitution method

Assume that we have the following SSDEs:

$$y_{t+1} = a_{11} y_t + a_{12} x_t + b_y, \quad \text{and} \quad x_{t+1} = a_{21} y_t + a_{22} x_t + b_x \tag{9.4.1}$$

In Sections 9.2.3 and 9.3.2 we solved the first-and-higher-order difference equations by finding and, then, combining the particular solutions and the complementary functions. But, to find both particular solution and complementary function we used trial solutions. The solution of a SSDE also involves two parts: the particular solution and the complementary function.

Let us first determine the particular solution to the system (9.4.1). As we used in Sections 9.2.3 and 9.3.2, we use trial solutions $y_t = p_{ys}$ and $x_t = p_{xs}$, where p_{ys} *and* p_{xs} are constants, which imply that $y_{t+1} = p_{ys}$ and $x_{t+1} = p_{xs}$. Substitution of these equations into the system (9.4.1) yields $p_{ys} = a_{11} p_{ys} + a_{12} p_{xs} + b_y$ and $p_{xs} = a_{21} p_{ys} + a_{22} p_{xs} + b_x$, which can be solved simultaneously to obtain the particular solutions

$$y_s = p_{ys} = \frac{(1-a_{22})b_y + a_{12}b_x}{(1-a_{11})(1-a_{22}) - a_{12}a_{21}} \quad \text{and} \tag{9.4.2}$$

$$x_s = p_{xs} = \frac{a_{21}b_x + (1-a_{11})b_y}{(1-a_{11})(1-a_{22}) - a_{12}a_{21}}. \tag{9.4.3}$$

In case the trial solutions $y_t = p_{ys}$ and $x_t = p_{xs}$ fail to work [that is, when the denominators in equations (9.4.2) and (9.4.3) happen to be zeros], then we have to use, as saw in Sections 9.2.3 and 9.3.2, trial solutions such as $y_t = p_{ys} t$ and $x_t = p_{xs} t$; $y_t = p_{ys} t^2$ and $x_t = p_{xs} t^2$; etc.

Having found out the particular solutions, we can now attempt to find out the complementary functions. For this we can use the homogenous version of the system in (9.4.1):

$$y_{t+1} = a_{11}y_t + a_{12}x_t, \quad \text{and} \quad x_{t+1} = a_{21}y_t + a_{22}x_t \tag{9.4.4}$$

We shall now advance the time in the first equation in the system (9.4.4) by one period to obtain $y_{t+2} = a_{11}y_{t+1} + a_{12}x_{t+1}$ and, then, substitute the second equation in the system into the last result, which yields $y_{t+2} = a_{11}y_{t+1} + a_{12}x_{t+1} = a_{11}y_{t+1} + a_{12}[a_{21}y_t + a_{22}x_t]$. Notice that the last equation still contains x_t, which can be obtained from the first equation in the system as

$$x_t = (y_{t+1} - a_{11}y_t)/a_{12}. \tag{9.4.5}$$

We can now substitute equation (9.4.5) into $y_{t+2} = a_{11}y_{t+1} + a_{12}[a_{21}y_t + a_{22}x_t]$ to yield $y_{t+2} = a_{11}y_{t+1} + a_{12}a_{21}y_t + a_{12}a_{22}[(y_{t+1} - a_{11}y_t)/a_{12}]$. Simplification of the last equation yields

$$y_{t+2} = (a_{11} + a_{22})y_{t+1} - (a_{11}a_{22} - a_{12}a_{21})y_t, \tag{9.4.6}$$

which is a second-order, linear, autonomous, homogenous difference equation in one variable (y). Notice that equation (9.4.6) is identical with equation (9.3.1) with $a_1 = a_{11} + a_{22}, a_2 = a_{11}a_{22} - a_{12}a_{21}, \quad and \ b = 0$. The reader must have noticed that when we moved from equation (9.4.4) to equation (9.4.6) we transformed a system of two, first-order, linear, autonomous, homogenous difference equation in two variables into a second-order, linear, autonomous, homogenous difference equation in one variable. This implies that we can, if we want to, go backward from equation (9.4.6) to equation (9.4.4). What all these mean is that we can always convert through substitution a simultaneous system of n first-order, linear, autonomous difference equations in n variables to an n$^{\text{th}}$-order, linear, autonomous, difference equation in one variable, and vice versa.

Let us now use the same method as that we used in Sections 9.2 and 9.3 to find the complementary function. Following that we obtain $v^2 - (a_{11} + a_{22})v + (a_{11}a_{22} - a_{12}a_{21}) = 0$, which is the characteristic equation of difference equation (9.4.6). Solving this equation yields the characteristic roots $v_1 \ and \ v_2$. If they are real and distinct, the complementary function becomes

$$y_c = C_1v_1^t + C_2v_2^t. \tag{9.4.7}$$

We now need to find the complementary function for x_t. This can be found by substituting equation (9.4.7) into equation (9.4.5). Once we do this and simplify, we obtain the complementary function for x_t as

$$x_c = \left[\frac{v_1 - a_{11}}{a_{12}}\right]C_1v_1^t + \left[\frac{v_2 - a_{11}}{a_{12}}\right]C_2v_2^t. \tag{9.4.8}$$

If the roots are real and repeated ($v = v_1 = v_2$), then the complementary function for y_t becomes

$$y_c = C_1v^t + C_2tv^t, \tag{9.4.9}$$

and the complementary function for x_t becomes

$$x_c = \left[\frac{(v - a_{11})(C_1 + C_2t)}{a_{12}}\right]v^t + \left[\frac{v}{a_{12}}\right]C_2v^t. \tag{9.4.10}$$

If the roots are complex, then the complementary function for y_t becomes

$$y_c = h^t[C_1\cos(\theta t) + C_2\sin(\theta t)], \tag{9.4.11}$$

and the complementary function for x_t becomes

$$x_c = h^t\left[\left(\frac{C_1 h\cos\theta + C_2 h\sin\theta - a_{11}C_1}{a_{12}}\right)\cos(\theta t)\right] + h^t\left[\left(\frac{C_2 h\cos\theta + C_1 h\sin\theta - a_{11}C_2}{a_{12}}\right)\sin(\theta t)\right]. \tag{9.4.12}$$

Let us now combine the particular solutions [equations (9.4.2) and (9.4.3)] with the complementary functions to obtain the complete solution. If the roots are real and distinct, the complete solutions are

$$y_t = y_c + y_s = C_1 v_1^t + C_2 v_2^t + y_s, \text{ and } x_t = x_c + x_s = \left[\frac{v_1 - a_{11}}{a_{12}}\right]C_1 v_1^t + \left[\frac{v_2 - a_{11}}{a_{12}}\right]C_2 v_2^t + x_s. \tag{9.4.13}$$

If the roots are real and repeated, the complete solutions are

$$y_t = y_c + y_s = C_1 v^t + C_2 t v^t + y_s, \text{ and } x_t = x_c + x_s = \left[\frac{(v - a_{11})(C_1 + C_2 t)}{a_{12}}\right]v^t + \left[\frac{v}{a_{12}}\right]C_2 v^t + x_s. \tag{9.4.14}$$

If the roots are complex, the complete solutions are

$$y_t = y_c + y_s = h^t[C_1\cos(\theta t) + C_2\sin(\theta t)] + y_s, \text{ and} \tag{9.4.15}$$

$$x_t = x_c + x_s = h^t\left[\left(\frac{C_1 h\cos\theta + C_2 h\sin\theta - a_{11}C_1}{a_{12}}\right)\cos(\theta t)\right] + h^t\left[\left(\frac{C_2 h\cos\theta + C_1 h\sin\theta - a_{11}C_2}{a_{12}}\right)\sin(\theta t)\right] + x_s.$$

As an example, consider the SSDEs

$$y_{t+1} = 0.75y_t + 0.1x_t + 10, \quad \text{and} \quad x_{t+1} = 0.2y_t + 0.75x_t + 3.33 \tag{9.4.16}$$

Notice that the system (9.4.16) is identical with the system (9.4.1) with $a_{11} = 0.75, a_{12} = 0.1, a_{21} = 0.2, a_{22} = 0.75, b_y = 10,$ *and* $b_x = 3.33$. Let us first find the particular solutions. For this we can use equations (9.4.2) and (9.4.3). Using these equations with $a_{11} = 0.75, a_{12} = 0.1, a_{21} = 0.2, a_{22} = 0.75, b_y = 10,$ *and* $b_x = 3.33$ we obtain the particular solutions $y_s = p_{ys} = 66.6$ and $x_s = p_{xs} = 66.6$.

We can now find the complementary functions. Let us first advance time in the first equation of the system (9.4.16) by one period to obtain $y_{t+2} = 0.75y_{t+1} + 0.1x_{t+1} + 10$. We shall now substitute the second equation in the system into the last result to obtain $y_{t+2} = 0.75y_{t+1} + 0.02y_t + 0.075x_t + 10.5$. We may substitute, into the last equation, for $x_t = 10y_{t+1} - 7.5y_t - 10$ from the first equation of the system and the result is $y_{t+2} = 1.5y_{t+1} - 0.5425y_t + 3$, which is identical with equation (9.4.6) with $(a_{11} + a_{22}) = 1.5$ and $(a_{11}a_{22} - a_{12}a_{21}) = 0.5425$. We shall now use the trial solution $y_t = Cv^t$, which implies that $y_{t+1} = Cv^{t+1}$ and $y_{t+2} = Cv^{t+2}$. Substituting these into the homogenous form $y_{t+2} = 1.5y_{t+1} - 0.5425y_t$ of the difference equation we can write it as $Cv^t[v^2 - 1.5v + 0.5425] = 0$. Since $Cv^t \neq 0$, we have $v^2 - 1.5v + 0.5425 = 0$ with the roots $v_1 = 0.6086$ and $v_2 = 0.8914$, which are real and distinct. Now, using equations (9.4.7) and (9.4.8), we can write the complementary functions for y_t as $y_c = C_1 v_1^t + C_2 v_2^t = C_1(0.6086)^t + C_2(0.8914)^t$ and for x_t as $x_c = -1.414C_1(0.6086)^t + 1.414C_2(0.8914)^t$. We can now combine the particular solutions with complementary functions to obtain the complete solutions. Thus, using equations (9.4.13), the complete solutions are $y_t = C_1(0.6086)^t + C_2(0.8914)^t + 66.6$ and $x_t = -1.414C_1(0.6086)^t + 1.414C_2(0.8914)^t + 66.6$. Assuming initial conditions $y_0 = 10$ *and* $x_0 = 20$, then we can obtain $C_1 = -11.82$ *and* $C_2 = -44.78$. Thus, we can write the complete and definite solutions as $y_t = -11.84(0.6086)^t - 44.82(0.8914)^t + 66.6$ and $x_t = 16.74(0.6086)^t - 63.39(0.8914)^t + 66.6$. The graphs of these solutions are illustrated in Figure 9.4.1. Notice that both variables converge to the common steady-state value of 66.6.

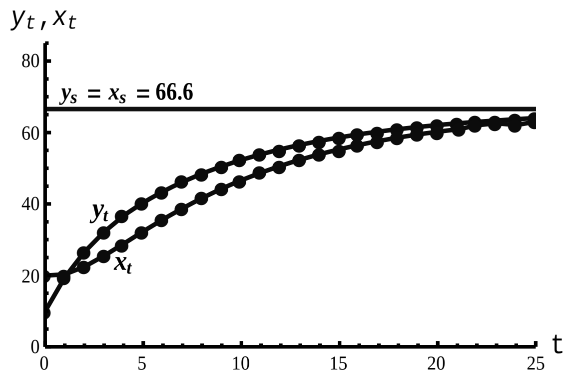


Figure 9.4.1

9.4.3 Solution of SSDEs: the matrix method

In the last section we found how we could solve the first order, linear, autonomous SSDEs in two variables by transforming them, using the substitution method, into second-order, linear, autonomous difference equations in one variable. But the substitution method will become more and more tedious as the number of equations and the number of variables in the system increase. Fortunately, there exists an alternative method that obviates the dependence on the substitution method. This alternative method is called the matrix method of solution to SSDEs and is more general in the sense that it can be applied to SSDEs that involve any number of equations and variables. We shall expose this method in the present section.

Suppose that we have the following general first-order, linear, autonomous SSDEs:

$$
\begin{aligned}
y_{1t} &= a_{11}y_{1t-1} + a_{12}y_{2t-1} + a_{13}y_{3t-1} + ... + a_{1n}y_{nt-1} + b_1 \\
y_{2t} &= a_{21}y_{1t-1} + a_{22}y_{2t-1} + a_{23}y_{3t-1} + ... + a_{2n}y_{nt-1} + b_2 \\
y_{3t} &= a_{31}y_{1t-1} + a_{32}y_{2t-1} + a_{33}y_{3t-1} + ... + a_{3n}y_{nt-1} + b_3 \,, \\
&\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \\
y_{nt} &= a_{n1}y_{1t-1} + a_{n2}y_{2t-1} + a_{n3}y_{3t-1} + ... + a_{nn}y_{nt-1} + b_n
\end{aligned}
\qquad (9.4.17)
$$

where $y_{it}, i = 1,2,3,...,n$, is the ith variable. Notice that the system (9.4.17) is difficult for one to solve with the substitution method. Therefore, we will use the matrix method. The system (9.4.17) can be written in matrix form as

$$\mathbf{Y_t} = \mathbf{AY_{t\text{-}1}} + \mathbf{B}, \qquad (9.4.18)$$

where $\mathbf{Y_t} = \begin{bmatrix} y_{1t} \\ y_{2t} \\ y_{3t} \\ \ldots \\ y_{nt} \end{bmatrix}$, $\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \ldots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \ldots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \ldots & a_{3n} \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ a_{n1} & a_{n2} & a_{n3} & \ldots & a_{nn} \end{bmatrix}$, $\mathbf{Y_{t-1}} = \begin{bmatrix} y_{1t-1} \\ y_{2t-1} \\ y_{3t-1} \\ \ldots \\ y_{nt-1} \end{bmatrix}$, and $\mathbf{B} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ \ldots \\ b_n \end{bmatrix}$. Let us first determine the particular solutions of y_{it} denoted by y_{is}. Notice that at this steady-state $y_{1s} = y_{1t-1}, y_{2s} = y_{2t-1}, y_{3s} = y_{3t-1}, \ldots, y_{ns} = y_{nt-1}$. Therefore, the system (9.4.17) in the steady-state becomes

$$\begin{aligned} y_{1s} &= a_{11}y_{1s} + a_{12}y_{2s} + a_{13}y_{3s} + \ldots + a_{1n}y_{ns} + b_1 \\ y_{2s} &= a_{21}y_{1s} + a_{22}y_{2s} + a_{23}y_{3s} + \ldots + a_{2n}y_{ns} + b_2 \\ y_{3s} &= a_{31}y_{1s} + a_{32}y_{2s} + a_{33}y_{3s} + \ldots + a_{3n}y_{ns} + b_3 \ , \\ &\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ y_{ns} &= a_{n1}y_{1s} + a_{n2}y_{2s} + a_{n3}y_{3s} + \ldots + a_{nn}y_{ns} + b_n \end{aligned} \tag{9.4.19}$$

which can be written in matrix form as

$$\mathbf{Y_s} = \mathbf{AY_s} + \mathbf{B}, \tag{9.4.20}$$

where $\mathbf{Y_s} = \begin{bmatrix} y_{1s} \\ y_{2s} \\ y_{3s} \\ \ldots \\ y_{ns} \end{bmatrix}$, $\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \ldots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \ldots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \ldots & a_{3n} \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ a_{n1} & a_{n2} & a_{n3} & \ldots & a_{nn} \end{bmatrix}$, and $\mathbf{B} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ \ldots \\ b_n \end{bmatrix}$. Equation (9.4.20) can be written as $\mathbf{Y_s} - \mathbf{AY_s} = \mathbf{B}$, $\mathbf{IY_s} - \mathbf{AY_s} = \mathbf{B}$, $(\mathbf{IY_s} - \mathbf{AY_s}) = \mathbf{B}$, or as$(\mathbf{I} - \mathbf{A})\mathbf{Y_s} = \mathbf{B}$. If the matrix $(\mathbf{I} - \mathbf{A})$ has an inverse [that is, if$(\mathbf{I} - \mathbf{A})^{-1}$ exists], then both sides of the last equation can be premultiplied by$(\mathbf{I} - \mathbf{A})^{-1}$ to obtain $(\mathbf{I} - \mathbf{A})^{-1}(\mathbf{I} - \mathbf{A})\mathbf{Y_s} = (\mathbf{I} - \mathbf{A})^{-1}\mathbf{B}$, which can be written as

$$\mathbf{Y_s} = (\mathbf{I} - \mathbf{A})^{-1}\mathbf{B}, \tag{9.4.21}$$

which gives us the particular solutions.

Let us now obtain the complementary functions. For this we use the homogenous version, of the system (9.4.17), given by

$$\begin{aligned} y_{1t} &= a_{11}y_{1t-1} + a_{12}y_{2t-1} + a_{13}y_{3t-1} + \ldots + a_{1n}y_{nt-1} \\ y_{2t} &= a_{21}y_{1t-1} + a_{22}y_{2t-1} + a_{23}y_{3t-1} + \ldots + a_{2n}y_{nt-1} \\ y_{3t} &= a_{31}y_{1t-1} + a_{32}y_{2t-1} + a_{33}y_{3t-1} + \ldots + a_{3n}y_{nt-1} \ , \\ &\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ y_{nt} &= a_{n1}y_{1t-1} + a_{n2}y_{2t-1} + a_{n3}y_{3t-1} + \ldots + a_{nn}y_{nt-1} \end{aligned} \tag{9.4.22}$$

which can be written in matrix form as

$$\mathbf{Y_{c,t}} = \mathbf{Y_t} = \mathbf{A}\ \mathbf{Y_{t-1}}, \tag{9.4.23}$$

where the subscript c shows that the system is concerned with the complementary function, or refers to the reduced form, of the system (9.4.17) and where

$\mathbf{Y_{c,t}} = \begin{bmatrix} y_{c,1t} \\ y_{c,2t} \\ y_{c,3t} \\ \ldots \\ y_{c,nt} \end{bmatrix}$, $\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \ldots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \ldots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \ldots & a_{3n} \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ a_{n1} & a_{n2} & a_{n3} & \ldots & a_{nn} \end{bmatrix}$, and $\mathbf{Y_{t-1}} = \begin{bmatrix} y_{1t-1} \\ y_{2t-1} \\ y_{3t-1} \\ \ldots \\ y_{nt-1} \end{bmatrix}$. Notice that equation (9.4.23) can be written as $\mathbf{Y_{t-1}} = \mathbf{AY_{t-2}}$. Therefore, equation (9.4.23) becomes $\mathbf{Y_{c,t}} = \mathbf{A(A)Y_{t-2}} = \mathbf{A^2Y_{t-2}}$. But, $\mathbf{Y_{t-2}} = \mathbf{AY_{t-3}}$ and, therefore, equation (9.4.23) becomes $\mathbf{Y_{c,t}} = \mathbf{A(A^2)Y_{t-3}} = \mathbf{A^3Y_{t-3}}$. One can continue like this until $t - t = 0$ yielding

$$\mathbf{Y_{c,t}} = \mathbf{A^t\, Y_0}, \tag{9.4.24}$$

where $\mathbf{Y_0}' = [y_{1,0} \quad y_{2,0} \quad y_{3,0} \quad \ldots \quad y_{n,0}]$; that is, $\mathbf{Y_0}$ is the column vector of the initial or $t = 0$ values of the variables. The power (t) may be a big value in certain applications. In that case, it is difficult to find $\mathbf{A^t}$. This compels us to diagonalize **A**, which we have learned to do in Section S2.8.7. Therefore, following equation (S2.8.33), we can write

$$\mathbf{A^t} = \mathbf{V\, D^t\, V^{-1}} = \mathbf{V} \begin{bmatrix} v_1^t & 0 & \ldots & 0 \\ 0 & v_2^t & \ldots & 0 \\ \ldots & \ldots & \ldots & 0 \\ 0 & 0 & \ldots & v_n^t \end{bmatrix} \mathbf{V^{-1}}, \tag{9.4.25}$$

where **D** represents the diagonal matrix whose diagonal elements $(v_1, v_2, \ldots, v_n)$ are the eigenvalues (or characteristic roots) of the coefficient matrix **A**, which can be found from the characteristic equation $|\mathbf{A} - v\,\mathbf{I}| = \mathbf{0}$, and **V** represents the matrix of the eigenvectors of matrix **A**. Using equation (9.4.25) we can rewrite equation (9.4.24) as

$$\mathbf{Y_{c,t}} = \mathbf{A^t\, Y_0} = \mathbf{V\, D^t\, V^{-1}\, Y_0} = \mathbf{V} \begin{bmatrix} v_1^t & 0 & \ldots & 0 \\ 0 & v_2^t & \ldots & 0 \\ \ldots & \ldots & \ldots & 0 \\ 0 & 0 & \ldots & v_n^t \end{bmatrix} \mathbf{V^{-1}\, Y_0}. \tag{9.4.26}$$

Let us now define $\mathbf{C} = \mathbf{V^{-1}Y_0}$, where $\mathbf{C}' = [C_1 \quad C_2 \quad \ldots \quad C_n]$, the vector of n constants. Therefore, equation (9.4.26) can be rewritten as

$$\mathbf{Y_{c,t}} = \mathbf{A^t\, Y_0} = \mathbf{V\, D^t\, V^{-1}\, Y_0} = \mathbf{V} \begin{bmatrix} v_1^t & 0 & \ldots & 0 \\ 0 & v_2^t & \ldots & 0 \\ \ldots & \ldots & \ldots & 0 \\ 0 & 0 & \ldots & v_n^t \end{bmatrix} \mathbf{C}, \tag{9.4.27}$$

which gives us the complementary functions. Combining equations (9.4.21) and (9.4.27) yields the complete solution to the SSDEs (9.4.17):

$$\mathbf{Y_t} = \mathbf{Y_{c,t}} + \mathbf{Y_s} = \mathbf{A^t\, Y_0} + \mathbf{(I - A)^{-1}\, B} = \mathbf{V\, D^t\, V^{-1}\, Y_0} + \mathbf{(I - A)^{-1}\, B} = \mathbf{V\, D^t\, C} + \mathbf{(I - A)^{-1}\, B}$$

$$= \mathbf{V} \begin{bmatrix} v_1^t & 0 & \ldots & 0 \\ 0 & v_2^t & \ldots & 0 \\ \ldots & \ldots & \ldots & 0 \\ 0 & 0 & \ldots & v_n^t \end{bmatrix} \mathbf{C} + \mathbf{(I - A)^{-1}\, B}. \tag{9.4.28}$$

As before, we can definitize the solution in equation (9.4.28) using initial conditions. Notice that the result in equation (9.4.28) is applicable when the eigenvalues or the characteristic roots are real and distinct. When the roots are real and equal or when they are complex numbers, we need to modify the result in equation (9.4.28). But, these modifications are beyond the scope of this book and, therefore, we omit them.

As an application of the matrix method, consider the problem we solved through the substitution method in the last section. The SSDEs in this example is given in equation (9.4.16). Notice that the system can be written, by moving time backward by one period, as

$$y_t = 0.75y_{t-1} + 0.1x_{t-1} + 10, \quad \text{and} \quad x_t = 0.2y_{t-1} + 0.75x_{t-1} + 3.33, \tag{9.4.29}$$

which is identical with the general system (9.4.17) if we treat $y_t = y_{1t}$, $y_{t-1} = y_{1t-1}$, $x_t = y_{2t}$, $x_{t-1} = y_{2t-1}$, $a_{11} = 0.75, a_{12} = 0.1, a_{21} = 0.2, a_{22} = 0.75$, $b_1 = 10,$ *and* $b_2 = 3.33$.

Let us first determine the particular solutions. In the steady-state the variables are such that $y_s = y_t = y_{t-1}$ and $x_s = x_t = x_{t-1}$. Therefore, the system (9.4.29) can be written, in the steady-state, as

$$y_s = 0.75y_s + 0.1x_s + 10, \quad \text{and} \quad x_s = 0.2y_s + 0.75x_s + 3.33 \tag{9.4.30}$$

which can be written in matrix form as equation (9.4.20) with $\mathbf{Y_s} = \begin{bmatrix} y_s \\ x_s \end{bmatrix}$, $\mathbf{A} = \begin{bmatrix} 0.75 & 0.1 \\ 0.2 & 0.75 \end{bmatrix}$, and $\mathbf{B} = \begin{bmatrix} 10 \\ 3.33 \end{bmatrix}$. We can now apply equation (9.4.21), $\mathbf{Y_s} = (\mathbf{I} - \mathbf{A})^{-1}\mathbf{B}$, to obtain the particular solutions. But, for this we first need to find $(\mathbf{I} - \mathbf{A})^{-1}$. Since $(\mathbf{I} - \mathbf{A}) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 0.75 & 0.1 \\ 0.2 & 0.75 \end{bmatrix} = \begin{bmatrix} 0.25 & -0.1 \\ -0.2 & 0.25 \end{bmatrix}$, $(\mathbf{I} - \mathbf{A})^{-1} = \begin{bmatrix} 5.88235 & 2.35294 \\ 4.70588 & 5.88235 \end{bmatrix}$. Therefore, applying equation (9.4.21) gives the particular solutions

$$\mathbf{Y_s} = (\mathbf{I} - \mathbf{A})^{-1}\mathbf{B} = \begin{bmatrix} 5.88235 & 2.35294 \\ 4.70588 & 5.88235 \end{bmatrix} \begin{bmatrix} 10 \\ 3.33 \end{bmatrix} = \begin{bmatrix} 66.6 \\ 66.6 \end{bmatrix} = \begin{bmatrix} y_s \\ x_s \end{bmatrix}. \tag{9.4.31}$$

Therefore, we have $y_s = x_s = 66.6$, which precisely were the particular solutions we obtained with the substitution method in the last section.

We can now determine the complementary functions. For this we can convert the system (9.4.29) into its homogenous form by treating the constants as zeros. This gives us the system

$$y_t = 0.75y_{t-1} + 0.1x_{t-1}, \quad \text{and} \quad x_t = 0.2y_{t-1} + 0.75x_{t-1}, \tag{9.4.32}$$

which can be written in matrix form as equation (9.4.23) or as equation (9.4.24) with $\mathbf{Y_{c,t}}' = [y_{c,t} \quad x_{c,t}]$. Now, following equation (9.4.27), we can write $\mathbf{Y_{c,t}} = \mathbf{A^t}\,\mathbf{Y_0} = \mathbf{V}\,\mathbf{D^t}\,\mathbf{V^{-1}}\,\mathbf{Y_0} = \mathbf{V}\,\mathbf{D^t}\,\mathbf{C}$, where $\mathbf{V}$ is the matrix of eigenvectors generated using the eigenvalues (v_1 *and* v_2) of the matrix $\mathbf{A}$. Using the characteristic equation $|\mathbf{A} - v\,\mathbf{I}| = \mathbf{0}$ we can find the eigenvalues of the matrix $\mathbf{A}$ as $v_1 = 0.6086$ *and* $v_2 = 0.8914$. Notice that these were precisely the eigenvalues we obtained in the last section. Then we can obtain $\mathbf{V} = \begin{bmatrix} 1 & 1 \\ -1.414 & 1.414 \end{bmatrix}$. Therefore, application of equation (9.4.27) yields $\mathbf{Y_{c,t}} = \mathbf{V}\,\mathbf{D^t}\,\mathbf{C} = \mathbf{V}\begin{bmatrix} v_1^t & 0 \\ 0 & v_2^t \end{bmatrix}\mathbf{C} = \begin{bmatrix} 1 & 1 \\ -1.414 & 1.414 \end{bmatrix}\begin{bmatrix} 0.6086^t & 0 \\ 0 & 0.8914^t \end{bmatrix}\begin{bmatrix} C_1 \\ C_2 \end{bmatrix}$

$$= \begin{bmatrix} C_1(0.6086)^t + C_2(0.8914)^t \\ C_1(-1.414)(0.6086)^t + C_2(1.414)(0.8914)^t \end{bmatrix} = \begin{bmatrix} y_{c,t} \\ x_{c,t} \end{bmatrix}. \tag{9.4.33}$$

Equation (9.4.33) gives the complementary functions. We shall now combine these complementary functions with the particular solutions in equation (9.4.31) to obtain the complete solution. Therefore, applying equation (9.4.28), the complete solution can be written as $\mathbf{Y_t} = \mathbf{Y_{c,t}} + \mathbf{Y_s} = \mathbf{V\,D^t\,C} + (\mathbf{I}-\mathbf{A})^{-1}\,\mathbf{B}$

$$=\begin{bmatrix} y_{c,t} \\ x_{c,t} \end{bmatrix}+\begin{bmatrix} y_s \\ x_s \end{bmatrix}=\begin{bmatrix} C_1(0.6086)^t + C_2(0.8914)^t \\ C_1(-1.414)(0.6086)^t + C_2(1.414)(0.8914)^t \end{bmatrix}+\begin{bmatrix} 66.6 \\ 66.6 \end{bmatrix}\text{, or} \tag{9.4.34}$$

as $y_t = C_1(0.6086)^t + C_2(0.8914)^t + 66.6$, and $x_t = -1.414C_1(0.6086)^t + 1.414C_2(0.8914)^t + 66.6$, which precisely were the complete solutions we obtained at the end of the last section.

9.4.4 SSDEs: convergence

In Sections 9.2 and 9.3 we discussed the convergence of the solution paths of the variables in the first-and-higher-order single difference equations. We found in both these discussions that convergence of the solutions to these equations depended solely on the absolute values of the eigenvalues or of the roots of the characteristic equations of the difference equations. Specifically, our result was that the solution path of the variable would converge to (diverge from) its intertemporal equilibrium value or the steady-state value if the absolute values of the roots were less (greater) than 1. This is valid irrespective of the order of the difference equation. But, as we saw in Section 9.3.6, obtaining the eigenvalues in the case of higher-order difference equations is a problem and, therefore, a quantitative analysis of convergence is often difficult. Even in this case we can carry out a qualitative analysis using Schur theorem outlined in Section 9.3.6.

One might wonder how we can get information on convergence in the case of SSDEs with which we were concerned so far in the present section. A close inspection of equation (9.4.28), or its specific case in equation (9.4.34), would suggest the clue. Notice that the movements of y_t *and* x_t in equation (9.4.34) are still dependent upon the terms in the brackets as they are the function of time. Since the values inside the brackets are less than one in absolute terms, and since they are raised to the power t, they must decline as t increases. Moreover, the values inside the brackets in equation (9.4.34) are the eigenvalues of the characteristic equation of the associated SSDEs.

The above discussion points to the fact that, as in the case of single linear difference equations, convergence of the time paths of variables in SSDEs are also dependent upon the eigenvalues of the characteristic equation of the system. If the absolute values of all the associated eigenvalues are less (greater) than one, then the time paths of the variables of the difference equation converges to (diverges from) the steady-state value. These were the reasons why the time paths of the variables x *and* y in the solutions in equation (9.4.34) converged to their respective and equal steady-state values as shown by the graphs in Figure 9.4.1.

9.4.5 Phase plane analysis of SSDEs

We exposed the phase diagram analyses of linear and nonlinear difference equations in Sections 9.2.5 and 9.2.9, respectively. In these sections our aim was to draw qualitative inferences about the time paths of the variables of the difference equations even without solving them explicitly. Our main inference was that the slope of the phase line, which is the graph of the value of the variable in one period as function of the value of the same variable in the immediate past period, determined whether the time path of the variable of the equation converged to or diverged from the steady-state value. We observed that if the absolute value of the slope of the phase line was in between 0 and 1, then the time path of the variable would converge to the steady-state. Other wise the time path would diverge from the steady-state.

If we could solve SSDEs explicitly, as the solution in equation (9.4.34), we could judge whether the time paths of the variables of the system would converge to or diverge from the steady-state using the absolute values of the associated eigenvalues. But, some SSDEs, whether linear or nonlinear, are difficult to be solved explicitly. Even in this event we can conduct a qualitative analysis of the nature of the time paths of the variables and this analysis is called **phase plane analysis** of SSDEs, which is the topic of the present section.

Let us illustrate this analysis with the same example of the two-variable, two-equation SSDEs (9.4.16): $x_{t+1} = 0.2y_t + 0.75x_t + 3.33$ and $y_{t+1} = 0.75y_t + 0.1x_t + 10$. We can transform this system by subtracting

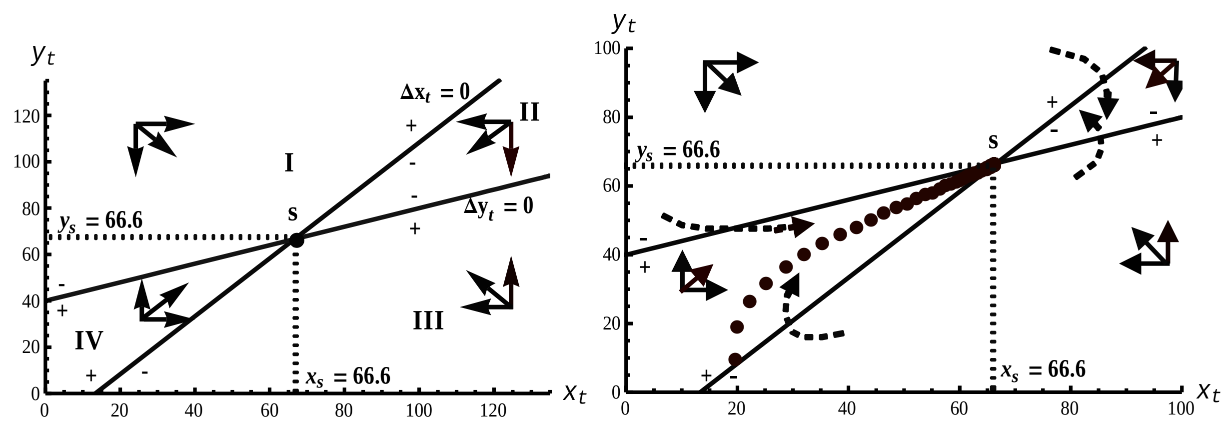
x_t *and* y_t (form both sides) of the first and the second equations, respectively, to obtain $\Delta x_t = x_{t+1} - x_t = 0.2y_t + 0.75x_t + 3.33 - x_t$ and $\Delta y_t = y_{t+1} - y_t = 0.75y_t + 0.1x_t + 10 - y_t$, which can be simplified to obtain $\Delta x_t = x_{t+1} - x_t = -0.25x_t + 0.2y_t + 3.33$ and $\Delta y_t = y_{t+1} - y_t = 0.1x_t - 0.25y_t + 10$, respectively. Notice that, in these equations, Δx_t *and* Δy_t are functions of x_t *and* y_t; that is, $\Delta x_t = f(x_t, y_t)$ and $\Delta y_t = g(x_t, y_t)$. Hence we have the following functions:

$$\Delta x_t = f(x_t, y_t) = -0.25x_t + 0.2y_t + 3.33 \quad \text{and} \quad \Delta y_t = g(x_t, y_t) = 0.1x_t - 0.25y_t + 10. \tag{9.4.35}$$

If $\Delta x_t = 0$, then the variable x_t is in equilibrium or at its steady-state. Similarly, if $\Delta y_t = 0$, then the variable y_t is in equilibrium or at its steady-state. Assume for the moment that $\Delta x_t = \Delta y_t = 0$. This helps us write the two equations in the system (9.4.35) as $\Delta x_t = 0 \Rightarrow f(x_t, y_t) = -0.25x_t + 0.2y_t + 3.33 = 0$ and $\Delta y_t = 0 \Rightarrow g(x_t, y_t) = 0.1x_t - 0.25y_t + 10 = 0$, which can be solved for y_t to obtain

$$\Delta x_t = 0 \Rightarrow y_t = -16.65 + 1.25x_t \quad \text{and} \quad \Delta y_t = 0 \Rightarrow y_t = 40 + 0.4x_t \tag{9.4.36}$$

The two equations in the system (9.4.36) are plotted in Panel (A) of Figure 9.4.2 and are denoted by $\Delta x_t = 0$ and $\Delta y_t = 0$ lines, respectively. These lines are called **demarcation lines** because each of them divides the total (x_t, y_t) plane or space into two phases or subplanes giving a total of four subplanes or phases denoted by I, II, III, and IV. These subplanes are called **phase planes** and an analysis of the behavior of the variables of a SSDEs using phase planes is referred to as phase plane analysis of difference equations.



(A) (B)

Figure 9.4.2

Notice few features of the demarcations lines in Panel (A) of Figure 9.4.2. The first is that both lines are straight lines because the associated first derivatives of y_t with respect to x_t in the system (9.4.36) are both constants: 1.25 and 0.4. The second is that these lines have different but positive slopes (1.25>0 and 0.4>0) suggesting that the lines lie upward from left to right. These imply that both demarcation lines increase as x_t increases. Moreover, since they are linear lines and with different slopes, the demarcation lines will cross each other at some point (s) in the (x_t, y_t) plane.

Let us now see how the phase planes can be used to analyze the behavior of the variables of a SSDEs. For this we can use the system (9.4.35). We stated above that the variables x *and* y are in state-state on $\Delta x_t = 0$ and $\Delta y_t = 0$ lines, respectively. We can find the impact of a change in $x_t(y_t)$ on $\Delta x_t(\Delta y_t)$ if we differentiate $\Delta x_t(\Delta y_t)$ in the system (9.4.35) partially with respect to $x_t(y_t)$. If $\partial \Delta x_t / \partial x_t > 0 (\partial \Delta y_t / \partial y_t > 0)$, then an increase in $x_t(y_t)$ increases $\Delta x_t(\Delta y_t)$ and, therefore, $x_t(y_t)$ increases eventually. If $\partial \Delta x_t / \partial x_t < 0 (\partial \Delta y_t / \partial y_t < 0)$, then an increase in $x_t(y_t)$ decreases $\Delta x_t(\Delta y_t)$ and, therefore, $x_t(y_t)$ decreases eventually. These suggest that if

$\Delta x_t (\Delta y_t)$ is positive when $x_t (y_t)$ changes, then $x_t (y_t)$ will increase. The opposite will happen if $\Delta x_t (\Delta y_t)$ is negative when $x_t (y_t)$ changes.

In our example, we obtain from the system (9.4.35) that $\partial \Delta x_t / \partial x_t = -0.25 < 0$. Since the partial derivative is negative, Δx_t must decrease (increase) as x_t increases (decreases) and, consequently x_t decreases (increases). Therefore, on the RHS of the demarcation line $\Delta x_t = 0$ we have a decrease in x_t. But, if x_t decreases on the RHS of the demarcation line $\Delta x_t = 0$, it must increase on the LHS of the line. In short, we have a decreasing x_t and an increasing x_t on the RHS and LHS, respectively, of the demarcation line $\Delta x_t = 0$. We have denoted these effects by minus sign and plus sign (or by leftward and rightward pointing arrows), respectively, in Panel (A) of Figure 9.4.2.

Similarly, we have from the system (9.4.35) $\partial \Delta y_t / \partial y_t = -0.25 < 0$. Since the partial derivative is negative, Δy_t must decrease (increase) as y_t increases (decreases) and, consequently y_t decreases (increases). Therefore, above the demarcation line $\Delta y_t = 0$ we have a decrease in y_t. But, if y_t decreases above the demarcation line $\Delta y_t = 0$, it must increase below the line. In short, we have a decreasing y_t and an increasing y_t above and below, respectively, of the demarcation line $\Delta y_t = 0$. We have denoted these effects by minus sign and plus sign (or by downward and upward pointing arrows), respectively, in Panel (A) of Figure 9.4.2. What all these mean are that both x_t *and* y_t increase in phase **IV**, both x_t *and* y_t decrease in phase **II**, y_t decreases and x_t increases in phase **I**, and y_t increases and x_t decreases in phase **III**.

It is now clear that, no matter where x_t *and* y_t begin with their initial values, they will eventually converge to the point s, the point that corresponds to their common and identical steady-state value. But, how do we find the values that correspond to the point s? Notice that the point s is common to both demarcation lines and these demarcation lines, as we mentioned above, are defined by the two equations in the system (9.4.35). This implies that the values that correspond to the point s can be found by solving the two simultaneous equations in the system. Therefore, the solution is $x_t = x_s = y_t = y_s = 66.6$.

We now know that the intertemporal behavior of the variables in a SSDEs can be determined qualitatively through the phase plane analysis. In our specific example above, we found that both variables converged to the common steady-state value. But, this was the same as the conclusion we obtained with Figure 9.4.1, which was based on the explicit solution of the same SSDEs in Section 9.4.2. The advantage of the phase plane analysis lies in the determination of the behavior of variables in a SSDEs that is difficult to be solved explicitly.

An astute reader might ask how the variables move if they happen to be, with initial values, somewhere in any one of the phases in Panel (A) of Figure 9.4.2. If we didn't solve the system explicitly, we could only conjecture how the variables move. They may start with initial values at a point in any one of the phases and may take any route, including those represented by the dashed arrows in Panel (B) of Figure 9.4.2. These arrows are called **trajectories** or **streamlines**. If we have already solved the system explicitly then we can use these solutions to construct a scatter diagram of the variables and, then, superimpose the scatter diagram on the graphs in Panel (A) of the figure. The result will give us the exact path of the movement or exact trajectory of the variables to the steady-state [as represented by the path of thick dots in Panel (B) of the same figure in the present example] or away from it. Notice that this path is also a trajectory. Notice also that one could obtain a variety of phase planes and, thereby, a variety of equilibria depending upon the forms of demarcation lines and the values of the associated derivatives.

9.4.6 Simultaneous systems of nonlinear difference equations

We considered single, nonlinear, autonomous difference equations in Section 9.2.9. We demonstrated there how to linearize such equations and, then, solve them. However, our discussion there was mainly centered on deriving the qualitative information about the convergence of the variable of the equation. This was due to the fact that it was difficult to solve many of the nonlinear difference equations explicitly. Our qualitative analysis showed that the absolute value of the first-order derivative of the phase line evaluated at the steady-state value determined the convergence or divergence of the variable. If this absolute value is less (greater) than one, then the variable of the difference equation will converge to (diverge from) its steady-state.

We saw in the last section how one could derive qualitative information about the intertemporal behavior of variables involved in a SSDEs. But, we know that it is much better if we can solve the system explicitly, which obviates the need for qualitative analysis. This is often achieved by **linearizing systems of nonlinear difference equations**. Let us now consider **simultaneous systems of nonlinear difference equations** (hereafter, **SSNDEs**) and see how we can solve them and, thereby, obtain information on the convergence of their variables. As an illustration, consider the system

$$x_t = f(x_{t-1}, y_{t-1}) \quad \text{and} \quad y_t = g(x_{t-1}, y_{t-1}), \tag{9.4.37}$$

where $f(x_{t-1}, y_{t-1})$ and $g(x_{t-1}, y_{t-1})$ are assumed to be nonlinear. We know that in the steady-state, if it exists, $x_{s-1} = x_s = x_{s+1}$ and $y_{s-1} = y_s = y_{s+1}$. Therefore, the system (9.4.37) can be written as

$$x_s = f(x_s, y_s) \quad \text{and} \quad y_s = g(x_s, y_s). \tag{9.4.38}$$

Let us now linearize the system (9.4.37) around x_s *and* y_s using Taylor series introduced in Section 3.6 to obtain

$$x_t - x_s = \frac{\partial f(x_s, y_s)}{\partial x_{t-1}}(x_{t-1} - x_s) + \frac{\partial f(x_s, y_s)}{\partial y_{t-1}}(y_{t-1} - y_s) \text{ and} \tag{9.4.39}$$
$$y_t - y_s = \frac{\partial g(x_s, y_s)}{\partial x_{t-1}}(x_{t-1} - x_s) + \frac{\partial g(x_s, y_s)}{\partial y_{t-1}}(y_{t-1} - y_s),$$

where we have omitted the remainder terms for convenience. Defining $a_{11} = \partial f(x_s, y_s)/\partial x_{t-1}, a_{12} = \partial f(x_s, y_s)/\partial y_{t-1}, a_{21} = \partial g(x_s, y_s)/\partial x_{t-1}, \text{ and } a_{22} = \partial g(x_s, y_s)/\partial y_{t-1}$, the system (9.4.39) can be written as

$$x_t - x_s = a_{11}(x_{t-1} - x_s) + a_{21}(y_{t-1} - y_s) \quad \text{and} \quad y_t - y_s = a_{21}(x_{t-1} - x_s) + a_{22}(y_{t-1} - y_s), \tag{9.4.40}$$

which can be written alternatively as

$$x_t = a_{11}x_{t-1} + a_{21}y_{t-1} + [(1 - a_{11})x_s - a_{21}y_s] \text{ and} \quad y_t = a_{21}x_{t-1} + a_{22}y_{t-1} + [-a_{21}x_s + (1 - a_{22})y_s], \tag{9.4.41}$$

which can be written in matrix form as

Y$_t$ = A Y$_{t-1}$ + B$_s$, (9.4.42)

where $\mathbf{Y_t} = \begin{bmatrix} x_t \\ y_t \end{bmatrix}$, $\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} = \begin{bmatrix} \frac{\partial f}{\partial x_{t-1}} & \frac{\partial f}{\partial y_{t-1}} \\ \frac{\partial g}{\partial x_{t-1}} & \frac{\partial g}{\partial y_{t-1}} \end{bmatrix}$, $\mathbf{Y_{t-1}} = \begin{bmatrix} x_{t-1} \\ y_{t-1} \end{bmatrix}$, and $\mathbf{B_s} = \begin{bmatrix} (1 - a_{11})x_s - a_{21}y_s \\ -a_{21}x_s + (1 - a_{22})y_s \end{bmatrix}$. Equation (9.4.42) is similar to equation (9.4.18) with solution in equation (9.4.28). Therefore, applying equation (9.4.28), we obtain the solution to the system (9.4.42) as $\mathbf{Y_t} = \mathbf{Y_{c,t}} + \mathbf{Y_s} = \mathbf{V}\,\mathbf{D^t}\,\mathbf{V^{-1}}\,\mathbf{Y_0} + (\mathbf{I}\text{-}\mathbf{A})^{-1}\,\mathbf{B_s}$, where $\mathbf{Y_{c,t}} = \mathbf{V}\,\mathbf{D^t}\,\mathbf{V^{-1}}\,\mathbf{Y_0}$ and $\mathbf{Y_s} = (\mathbf{I}\text{-}\mathbf{A})^{-1}\,\mathbf{B_s}$. We can now define $\mathbf{C'} = [C_1 \quad C_2] = \mathbf{V^{-1}}\,\mathbf{Y_0}$, where $\mathbf{Y_0}' = [x_0 \quad y_0]$ and **V** represents the matrix of eigenvectors of **A**. Finally, we can write the solution to the system (9.4.41) as

$$\mathbf{Y_t} = \mathbf{Y_{c,t}} + \mathbf{Y_s} = \mathbf{V}\,\mathbf{D^t}\,\mathbf{V^{-1}}\,\mathbf{Y_0} + (\mathbf{I}\text{-}\mathbf{A})^{-1}\,\mathbf{B_s} = \mathbf{V}\,\mathbf{D^t}\,\mathbf{C} + (\mathbf{I}\text{-}\mathbf{A})^{-1}\,\mathbf{B_s}. \tag{9.4.43}$$

Notice that the solution (9.4.43) is the linearized solution to the nonlinear system (9.4.37). This implies that both solutions [the solution to the linearized system (9.4.43) and the solution to the original nonlinear system (9.4.37)] exhibit similar behavior for the variables x_t *and* y_t, particularly when the trace and the determinant of the Jacobian matrix **A** are non-zeroes.

9.4.7 Application examples

Example 1: Suppose that a variant of the model of expectations-augmented Phillips curve with adaptive expectations and monetary policy we solved in the third example in Section 9.3.7 is given by the linear SSDEs $p^e_{t+1} = 0.05p^e_t + 0.9u_t + 0.25$ and $u_{t+1} = 0.9p^e_t + 0.05u_t + 0.25$, where p^e *and* u denote expected rate of inflation and unemployment rate, respectively. Solve this SSDEs and determine whether the expected rate of inflation and unemployment rate tend to their identical or distinct steady-states (if they exists) or deviate from them using the initial conditions $p^e_0 = 4$ *and* $u_0 = 2$. Construct a phase plane for the model and determine the movement of the trajectory of the expected rate of inflation and the unemployment rate.

Solution: First of all, we can convert the two difference equations into the forms $p^e_t = 0.05p^e_{t-1} + 0.9u_{t-1} + 0.25$ and $u_t = 0.9p^e_{t-1} + 0.05u_{t-1} + 0.25$. Although we can use either the substitution or the matrix method, we shall use the matrix method to solve this problem. Following equation (9.4.18) we can write this system in matrix form as $\mathbf{Y_t} = \mathbf{A}\ \mathbf{Y_{t-1}} + \mathbf{B}$, where $\mathbf{Y_t} = \begin{bmatrix} p^e_t \\ u_t \end{bmatrix}$, $\mathbf{A} = \begin{bmatrix} 0.05 & 0.9 \\ 0.9 & 0.05 \end{bmatrix}$, $\mathbf{Y_{t-1}} = \begin{bmatrix} p^e_{t-1} \\ u_{t-1} \end{bmatrix}$, and $\mathbf{B} = \begin{bmatrix} 0.25 \\ 0.25 \end{bmatrix}$. Applying equation (9.4.21), we obtain the particular solution as $\mathbf{Y_s} = \begin{bmatrix} p^e_s \\ u_s \end{bmatrix} = (\mathbf{I} - \mathbf{A})^{-1}\ \mathbf{B} = \begin{bmatrix} 0.95 & -0.9 \\ -0.9 & 0.95 \end{bmatrix}^{-1} \begin{bmatrix} 0.25 \\ 0.25 \end{bmatrix} = \begin{bmatrix} 10.27 & 9.73 \\ 9.73 & 10.27 \end{bmatrix} \begin{bmatrix} 0.25 \\ 0.25 \end{bmatrix} = \begin{bmatrix} 5 \\ 5 \end{bmatrix}$, where $p^e_s = 5$ *and* $u_s = 5$ represent the steady-state values of expected inflation rate and unemployment rate respectively.

We can now determine the complementary functions for which we shall use the homogenous version of the difference equations: $p^e_t = 0.05p^e_{t-1} + 0.9u_{t-1}$ and $u_t = 0.9p^e_{t-1} + 0.05u_{t-1}$. Then, applying equation (9.4.26), we obtain $\mathbf{Y_{c,t}} = \mathbf{A^t}\,\mathbf{Y_0} = \mathbf{V}\ \mathbf{D^t}\ \mathbf{V^{-1}}\,\mathbf{Y_0} = \begin{bmatrix} 0.707 & 0.707 \\ 0.707 & -0.707 \end{bmatrix} \begin{bmatrix} (0.95)^t & 0 \\ 0 & (-0.85)^t \end{bmatrix} \mathbf{V^{-1}}\,\mathbf{Y_0}$, where $\mathbf{Y_0}$ represents the vector of initial values: $\mathbf{Y_0}' = [p^e_0 \quad u_0] = [4 \quad 2]$. Denoting $\mathbf{C}' = [C_1 \quad C_2] = \mathbf{V^{-1}Y_0}$, we obtain $\mathbf{Y_{c,t}} = \begin{bmatrix} p^e_{c,t} \\ u_{c,t} \end{bmatrix} = \begin{bmatrix} 0.707 & 0.707 \\ 0.707 & -0.707 \end{bmatrix} \begin{bmatrix} (0.95)^t & 0 \\ 0 & (-0.85)^t \end{bmatrix} \begin{bmatrix} C_1 \\ C_2 \end{bmatrix} = \begin{bmatrix} C_1 0.707(0.95)^t + C_2 0.707(-0.85)^t \\ C_1 0.707(0.95)^t - C_2 0.707(-0.85)^t \end{bmatrix}$. Therefore, the complementary functions are $p^e_{c,t} = C_1 0.707(0.95)^t + C_2 0.707(-0.85)^t$ and $u_{c,t} = C_1 0.707(0.95)^t - C_2 0.707(-0.85)^t$. Let us combine the particular solution with the complementary functions, using equation (9.4.28), $\mathbf{Y_t} = \mathbf{Y_{c,t}} + \mathbf{Y_s}$, to get the complete solution $\mathbf{Y_t} = \mathbf{Y_{c,t}} + \mathbf{Y_s} = \begin{bmatrix} p^e_t \\ u_t \end{bmatrix} = \begin{bmatrix} C_1 0.707(0.95)^t + C_2 0.707(-0.85)^t \\ C_1 0.707(0.95)^t - C_2 0.707(-0.85)^t \end{bmatrix} + \begin{bmatrix} 5 \\ 5 \end{bmatrix}$, or $p^e_t = C_1 0.707(0.95)^t + C_2 0.707(-0.85)^t + 5$ and $u_t = C_1 0.707(0.95)^t - C_2 0.707(-0.85)^t + 5$. We can now use the initial conditions $p^e_0 = 4$ *and* $u_0 = 2$ to get the values of C_1 *and* C_2 as $C_1 = -2.83$ *and* $C_2 = 1.42$. With these values of C_1 *and* C_2, the definite paths of p^e_t *and* u_t can be written as $p^e_t = -2.83 \times 0.707(0.95)^t + 1.42 \times 0.707(-0.85)^t + 5 = -2(0.95)^t + (-0.85)^t + 5$ and $u_t = -2.83 \times 0.707(0.95)^t - 1.42 \times 0.707(-0.85)^t + 5 = -2(0.95)^t - (-0.85)^t + 5$, respectively. The graphs of these two solutions are illustrated in Panel (A) of Figure 9.4.3, which exhibit that both the rate of expected inflation and the rate of unemployment converge to their common steady-state value (5).

Let us now determine the trajectories, in the associated phase plane, that represent the movements of the rates of expected inflation and unemployment. For this we proceed exactly as we did in Section 9.4.5. Firstly, we can take the first difference of the difference equations to obtain $\Delta p^e_t = p^e_t - p^e_{t-1} = -0.95p^e_{t-1} + 0.9u_{t-1} + 0.25$ and $\Delta u_t = u_t - u_{t-1} = 0.9p^e_{t-1} - 0.95u_{t-1} + 0.25$. When $\Delta p^e_t = 0$ *and* $\Delta u_t = 0$, we have $p^e_{t-1} = -0.2777 + 0.9u_{t-1}$ and $p^e_{t-1} = 0.2631 + 0.9474u_{t-1}$, or $p^e_t = -0.2777 + 1.0555u_t$ and $p^e_t = 0.2631 + 0.9474u_t$, respectively, which are plotted to obtain the demarcation lines (straight lines) in Panel (B) of Figure 9.4.3. Notice that $\partial\Delta p^e_t / \partial p^e_{t-1} = -0.95 < 0$ and

$\partial \Delta u_t / \partial u_{t-1} = -0.95 < 0$, which imply that both p_t^e and u_t decrease (increase) above (below) and on the right (left) of demarcation lines $\Delta p_t^e = 0$ and $\Delta u_t = 0$, respectively. These increase and decrease are shown by the positive and negative signs, respectively. Moreover, $dp_t^e / du_t \big|_{\Delta p_t^e = 0} = 1.0555 > 0$ and $dp_t^e / du_t \big|_{\Delta u_t = 0} = 0.9474 > 0$ implying that both demarcation lines increase from left to right. The last two equations also imply that the demarcation lines will cross each other only at one point, and that point is the steady-state or equilibrium point s with values $p_s^e = u_s = 5$. The last step is to plot the scatter diagram of p_t^e *and* u_t from the above solutions and superimpose it on the graphs of the demarcation lines which yield the required trajectories (the curves with heavy dots) as shown in Panel (B) of the same figure. As can be seen from the figure, the trajectories tend to the equilibrium point $s = p_s^e = u_s = 5$.

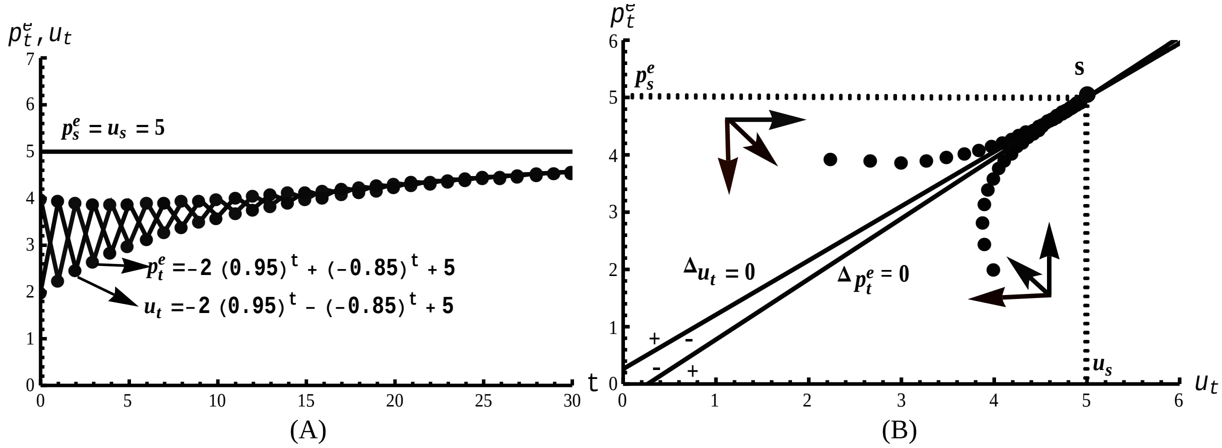


Figure 9.4.3

Example 2: Consider the problem in the dynamic Cournot duopoly example we solved in example 4 in Section 9.3.7. In that solution we derived the demand function the two firms y *and* x faced as $y_{t+2} = 50 - 0.5x_{t+1}$ and $x_{t+2} = 50 - 0.5y_{t+1}$, respectively. Solve these difference equations as a SSDEs assuming that $y_0 = 50$ *and* $x_0 = 0$.

Solution: As before, we shall first convert the two difference equations into the forms $y_t = 0y_{t-1} - 0.5x_{t-1} + 50$ and $x_t = -0.5y_{t-1} + 0x_{t-1} + 50$. Although we can use either the substitution or the matrix methods, we shall use the matrix method to solve this problem. Following equation (9.4.18) we can write this system in matrix form as $\mathbf{Y_t} = \mathbf{A}\ \mathbf{Y_{t-1}} + \mathbf{B}$, where $\mathbf{Y_t} = \begin{bmatrix} y_t \\ x_t \end{bmatrix}$, $\mathbf{A} = \begin{bmatrix} 0 & -0.5 \\ -0.5 & 0 \end{bmatrix}$, $\mathbf{Y_{t-1}} = \begin{bmatrix} y_{t-1} \\ x_{t-1} \end{bmatrix}$, and $\mathbf{B} = \begin{bmatrix} 50 \\ 50 \end{bmatrix}$. Applying equation (9.4.21), we obtain the particular solution $\mathbf{Y_s} = \begin{bmatrix} y_s \\ x_s \end{bmatrix} = (\mathbf{I} - \mathbf{A})^{-1}\ \mathbf{B} = \begin{bmatrix} 1 & 0.5 \\ 0.5 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 50 \\ 50 \end{bmatrix} = \begin{bmatrix} 1.333 & -0.666 \\ -0.666 & 1.333 \end{bmatrix} \begin{bmatrix} 50 \\ 50 \end{bmatrix}$ $= \begin{bmatrix} 33.33 \\ 33.33 \end{bmatrix}$, where $y_s = 33.33$ *and* $x_s = 33.33$ represent the steady-state values of outputs produced by the duopolists y *and* x, respectively. Notice that these were the same particular solutions as those we obtained in Section 9.3.7.

We shall now determine the complementary functions for which we shall use the homogenous version of the difference equations: $y_t = 0y_{t-1} - 0.5x_{t-1}$ and $x_t = -0.5y_{t-1} + 0x_{t-1}$. Then, applying equation (9.4.26), we obtain $\mathbf{Y_{c,t}} = \mathbf{A^t Y_0} = \mathbf{V\ D^t\ V^{-1}\ Y_0} = \mathbf{V} \begin{bmatrix} (0.5)^t & 0 \\ 0 & (-0.5)^t \end{bmatrix} \mathbf{V^{-1}\ Y_0} = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} (0.5)^t & 0 \\ 0 & (-0.5)^t \end{bmatrix} \mathbf{V}^{-1}\ \mathbf{Y_0}$, where $\mathbf{Y_0}$ represents the vector of initial values: $\mathbf{Y_0}' = [y_0 \quad x_0] = [50 \quad 0]$. Denoting $\mathbf{C}' = [C_1 \quad C_2] = \mathbf{V^{-1}\ Y_0}$, we obtain $\mathbf{Y_{c,t}} = \begin{bmatrix} y_{c,t} \\ x_{c,t} \end{bmatrix} =$

$\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} (0.5)^t & 0 \\ 0 & (-0.5)^t \end{bmatrix}\begin{bmatrix} C_1 \\ C_2 \end{bmatrix} = \begin{bmatrix} C_1(0.5)^t + C_2(-0.5)^t \\ -C_1(0.5)^t + C_2(-0.5)^t \end{bmatrix}$. Therefore, the complementary functions are $y_{c,t} = C_1(0.5)^t + C_2(-0.5)^t$ and $x_{c,t} = -C_1(0.5)^t + C_2(-0.5)^t$. Let us now combine the particular solution with the complementary functions, using equation (9.4.28), $\mathbf{Y_t} = \mathbf{Y_{c,t}} + \mathbf{Y_s}$, to get the complete solution $\mathbf{Y_t} = \mathbf{Y_{c,t}} + \mathbf{Y_s}$ $= \begin{bmatrix} y_t \\ x_t \end{bmatrix} = \begin{bmatrix} C_1(0.5)^t + C_2(-0.5)^t \\ -C_1(0.5)^t + C_2(-0.5)^t \end{bmatrix} + \begin{bmatrix} 33.33 \\ 33.33 \end{bmatrix}$, or $y_{c,t} = C_1(0.5)^t + C_2(-0.5)^t + 33.33$ and $x_{c,t} = -C_1(0.5)^t + C_2(-0.5)^t + 33.33$. We can now use the initial conditions $y_0 = 50$ *and* $x_0 = 0$ to get the values of C_1 *and* C_2 as $C_1 = 25$ *and* $C_2 = -8.33$. With these values of C_1 *and* C_2, the definite paths of y_t *and* x_t can be written as $y_t = 25(0.5)^t - 8.33_2(-0.5)^t + 33.33$ and $x_t = -25(0.5)^t - 8.33_2(-0.5)^t + 33.33$, respectively. The graphs of these two functions are illustrated in Panel (A) of Figure 9.4.4. Notice that, as expected, both solutions converge to the common steady-state value of 33.33.

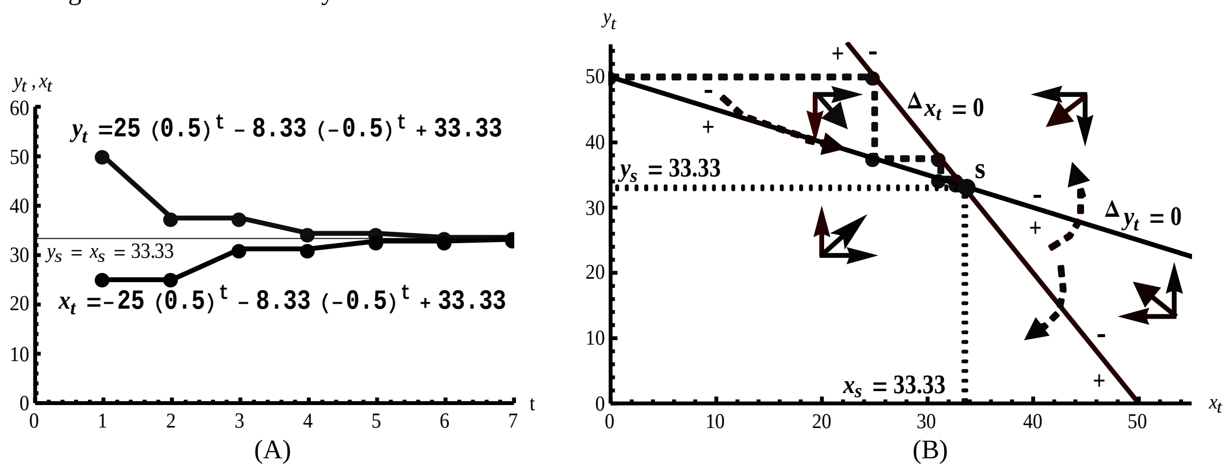


Figure 9.4.4

Let us now determine the trajectory, in the associated phase plane, that represents the movements of the quantities produced by the duopolists. For this we proceed exactly as we did in Section 9.4.5. Firstly, we can take the first difference of the difference equations to obtain $\Delta y_t = y_{t+1} - y_t = 50 - 0.5x_t - y_t$ and $\Delta x_t = x_{t+1} - x_t = 50 - 0.5y_t - x_t$. When $\Delta y_t = 0$ *and* $\Delta x_t = 0$, we have $y_t = 50 - 0.5x_t$ and $y_t = 100 - 2x_t$, respectively, which are plotted to obtain the demarcation lines (straight lines) in Panel (B) of Figure 9.4.4. Notice that $\partial \Delta y_t / \partial y_t = -1 < 0$ and $\partial \Delta x_t / \partial x_t = -1 < 0$ implying that both y_t *and* x_t decrease (increase) above (below) $\Delta y_t = 0$ and $\Delta x_t = 0$ lines. These decrease and increase are shown by the negative signs (or leftward and downward pointing arrows) and positive signs (or forward and rightward pointing arrows), respectively. Moreover, $dy_t / dx_t \big|_{\Delta y_t = 0} = -0.5 < 0$ and $dy_t / dx_t \big|_{\Delta x_t = 0} = -2 < 0$ implying that both demarcation lines decrease from left to right. The last two equations also imply that the demarcation lines will cross each other only at one point, and that point is the steady-state or equilibrium point s with values $y_s = x_s = 33.33$. The last step is to plot the scatter diagram of y_t *and* x_t and superimpose it on the graph of the demarcation lines which yields the required trajectory (the dashed line) as shown in Panel (B) of the figure. Notice that the trajectory tends to the equilibrium point.

Example 3: Suppose that the total output produced in period t (Y_t) in an economy using physical capital (K_t), human capital (H_t), labor (L_t) and technology (A_t) is given by the Cobb-Douglas production function $Y_t = K_t^{\alpha} H_t^{\beta} (A_t L_t)^{1-\alpha-\beta}$, where α *and* β denote the output elasticities of physical and human capital, respectively; and $\alpha > 0, \beta > 0,$ *and* $\alpha + \beta < 1$. Total saving (s) is assumed to be constant and is invested in physical and human capital with constant fractions s_K *and* s_H, respectively. This implies that physical and human capitals grow over time at the rates $\Delta K_t / K_t = (K_{t+1} - K_t) / K_t = s_K Y_t$ and $\Delta H_t / H_t = (H_{t+1} - H_t) / H_t = s_H Y_t$, respectively. Labor and technology are assumed to grow over time at constant rates n *and* g, respectively: $\Delta L_t / L_t = (L_{t+1} - L_t) / L_t = n$ and $\Delta A_t / A_t = (A_{t+1} - A_t) / A_t = g$. Lastly, assume that $y_t = Y_t / (A_t L_t); k_t = K_t / (A_t L_t);$

and $h_t = H_t/(A_t L_t)$, where $(A_t L_t)$ is referred to as ***effective labor***. Derive the SSDEs in k *and* h and show that k_t *and* h_t converge to their steady-state values k_s *and* h_s.

Solution: This is an example of the discrete time version of **Mankiw-Romer-Weil model** of economic growth. Given the above definitions, we can derive (following a procedure similar to the one we used in example 7 in Section 9.2.10) that $y_t = k_t^{\alpha} h_t^{\beta}$, $\Delta k_t = s_K y_t - (n+g)k_t$ and $\Delta h_t = s_H y_t - (n+g)h_t$, which represent the output, change in physical capital, and change in human capital per unit of effective labor, respectively. Substituting $y_t = k_t^{\alpha} h_t^{\beta}$ into the last two equations yields $\Delta k_t = s_K k_t^{\alpha} h_t^{\beta} - (n+g)k_t$ and $\Delta h_t = s_H k_t^{\alpha} h_t^{\beta} - (n+g)h_t$. The last two equations are the required SSDEs. Notice that these difference equations are nonlinear. The intertemporal behavior of the variables of these difference equations can be inferred either through linearization or through phase plane analysis. As it is much simpler, we shall choose the phase plane analysis to assess the said intertemporal behavior.

Let us now set the two difference equations to zero and solve them for k_t. That is, when $\Delta k_t = 0$, we have $k_t = [s_K/(n+g)]^{1/(1-\alpha)} h_t^{\beta/(1-\alpha)}$. The last equation gives the demarcation curve defined by $\Delta k_t = 0$. Similarly, when $\Delta h_t = 0$, we have $k_t = [(n+g)/s_H]^{1/\alpha} h_t^{(1-\beta)/\alpha}$, which gives the demarcation curve defined by $\Delta h_t = 0$. The first-derivative of the demarcation curve $k_t = [s_K/(n+g)]^{1/(1-\alpha)} h_t^{\beta/(1-\alpha)}$ with respect to h_t is $dk_t/dh_t = \beta[s_K/(n+g)]^{1/(1-\alpha)} h_t^{\beta/(1-\alpha)-1} > 0$ (because $\beta, s_K, n, g, h_t > 0$) and the second-derivative is $d^2 k_t/dh_t^2 = [\{\beta/(1-\alpha)\}-1]\beta[s_K/(n+g)]^{1/(1-\alpha)} h_t^{\beta/(1-\alpha)-2} < 0$ (because $[\{\beta/(1-\alpha)\}-1] < 0$). These derivatives imply that the demarcation curve defined by $\Delta k_t = 0$ increases at a diminishing rate in the phase plane. Similarly, the first-derivative of the demarcation curve $k_t = [(n+g)/s_H]^{1/\alpha} h_{t/}^{(1-\beta)/\alpha}$ with respect to h_t is $dk_t/dh_t = [(1-\beta)/\alpha][(n+g)/s_H]^{1/\alpha} h_{t/}^{((1-\beta)/\alpha)-1} > 0$ (because $(1-\beta), \alpha, n, g, s_H, h_t > 0$) and the second-derivative is $d^2 k_t/dh_t^2 = [\{(1-\beta)/\alpha\}-1][(1-\beta)/\alpha][s_K/(n+g)]^{1/(1-\alpha)} h_t^{\beta/(1-\alpha)-2} > 0$ (because $(1-\beta)/\alpha, \alpha, n, g, s_H, h_t > 0$). These imply that demarcation curve defined by $\Delta h_t = 0$ increases at an increasing rate in the phase plane. These curves are illustrated in Figure 9.4.5. Notice that these curves will intersect only at one point (s), except at 0, in the phase plane.

Since $\partial \Delta k_t / \partial k_t = \alpha s_K k_t^{\alpha-1} h_t^{\beta} - (n+g)$ and since $0 < \alpha < 1$, Δk_t decreases (increases), for given h_t, as k_t is above (below) the demarcation line $\Delta k_t = 0$. This implies that k_t decreases (increases) above (below) the demarcation line $\Delta k_t = 0$. Similarly, since $\partial \Delta h_t / \partial h_t = \beta s_K k_t^{\alpha} h_t^{\beta-1} - (n+g)$ and since $0 < \beta < 1$, Δh_t decreases (increases), for given k_t, as h_t is on the right (left) of the demarcation line $\Delta h_t = 0$. This implies that h_t decreases (increases) to the right (left) of the demarcation line $\Delta h_t = 0$. These increases and decreases in the variables are represented in the figure by the positive signs (or upward and rightward pointing arrows) and negative signs (or downward and leftward pointing arrows), respectively.

The last task in the present example is to determine the steady state-state values of the variables. These can be found by solving the SSDEs $\Delta k_t = s_K k_t^{\alpha} h_t^{\beta} - (n+g)k_t$ and $\Delta h_t = s_H k_t^{\alpha} h_t^{\beta} - (n+g)h_t$ when $\Delta k_t = 0$ and $\Delta h_t = 0$, respectively. Therefore, setting $\Delta k_t = s_K k_t^{\alpha} h_t^{\beta} - (n+g)k_t$ and $\Delta h_t = s_H k_t^{\alpha} h_t^{\beta} - (n+g)h_t$ to zero and solving using logarithms yields the steady-state values of physical and human capital per unit of effective labor as $k_s = \left[\left(s_K^{1-\beta} s_H^{\beta}\right)/(n+g)\right]^{1/(1-\alpha-\beta)}$ and $h_s = \left[\left(s_K^{\alpha} s_H^{1-\alpha}\right)/(n+g)\right]^{1/(1-\alpha-\beta)}$, respectively. Since $y_t = k_t^{\alpha} h_t^{\beta}$, output per unit of effective labor must also approach its steady-state (y_s) as k_t *and* h_t approach k_s *and* h_s: $y_s = k_s^{\alpha} h_s^{\beta}$. This steady-state value of y_t can be found by substituting k_s and h_s obtained above into $y_s = k_s^{\alpha} h_s^{\beta}$: $y_s = \left[\left(s_K^{1-\beta} s_H^{\beta}\right)/(n+g)\right]^{\alpha/(1-\alpha-\beta)} \times \left[\left(s_K^{\alpha} s_H^{1-\alpha}\right)/(n+g)\right]^{\beta/(1-\alpha-\beta)}$. As can be seen in Figure 9.4.5, both k_t *and* h_t approach or converge to their steady-state values no matter where they begin with initial values in the phase plane and, therefore, the trajectories will tend to the steady-state wherever k_t *and* h_t they begin. Notice that, since we did not solve the given SSDEs explicitly as they are nonlinear, we have not drawn the exact trajectory.

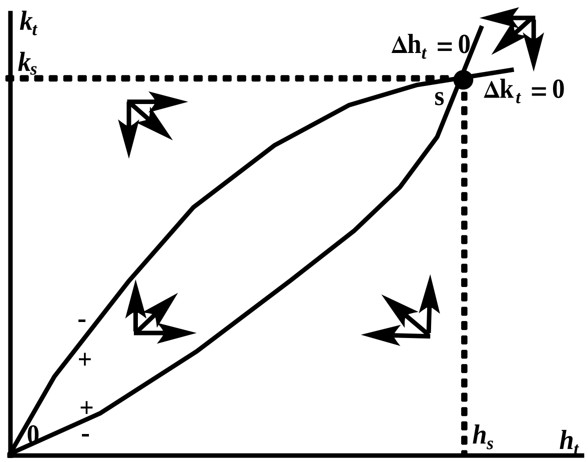


Figure 9.4.5

9.4.8 Exercises

1. Solve the following SSDEs assuming that the constants are equal to one and determine their exact trajectories in the associated phase planes.

(i) $y_t = 0.5x_{t-1} + 0.25y_{t-1} + 10$ and $x_t = 0.25x_{t-1} + 0.5y_t + 10$

(ii) $y_t = 0.25x_{t-1} + 0.5y_{t-1} + 5$ and $x_t = 0.5x_{t-1} + 0.25y_t + 5$.

2. **Application exercise**. Consider a variant of the inflation-unemployment problem solved in example 1 in Section 9.4.7. Solve the system, assuming that the constants are equal to one, and determine the exact trajectory of the variables in a phase plane if the SSDEs comprises of

$p_t^e = 0.2p_t^e + 0.8u_t$ and $u_t = 0.8p_t^e + 0.1u_t$.

3. **Application exercise**. Consider a variant of the dynamic Cournot duopoly problem solved in example 2 Section 9.4.7. Solve the system (assuming that $y_0 = 50$ *and* $x_0 = 0$ when $t = 0$) and determine the exact trajectory of the variables in a phase plane if the SSDEs comprises of

$y_{t+1} = 50 - 0.25x_t$ and $x_{t+1} = 50 - 0.25y_t$.

9.4.9 Mathematica applications

Mathematica's **RSolve** introduced in Sections 9.2 and 9.3 can also be used to solve SSDEs. As a demonstration, consider the system in 9.4.16. This SSDEs can be solved as

```
RSolve[{y[t+1]==0.75y[t]+0.1x[t]+10,0.2y[t]==-0.75x[t]+x[t+1]-
3.33,y[0]==10,x[0]==20},{y[t],x[t]},t]
```

```
{{y[t]→66.6588 -11.8372 0.608579^t-44.8216 0.891421^t,
x[t]→66.6471 +16.7403 0.608579^t-63.3874 0.891421^t}}
```